Casimir energy, the cosmological constant and massive gravitons

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The cosmological constant appearing in the Wheeler–De Witt equation is considered as an eigenvalue of the associated Sturm–Liouville problem. A variational approach with Gaussian trial wave functionals is used as a method to study such a problem. We approximate the equation to one loop in a Schwarzschild background and a zeta function regularization is involved to handle with divergences. The regularization is closely related to the subtraction procedure appearing in the computation of Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation. The case of massive gravitons is discussed.

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1 Introduction

One of the most fascinating and unsolved problems of the theoretical physics of our century is the cosmological constant. Einstein introduced his cosmological constant Λ_c in an attempt to generalize his original field equations. The modified field equations are

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where Λ_c is the cosmological constant, G is the gravitational constant and $T_{\mu\nu}$ is the energy–momentum tensor. By redefining

$$T_{\mu\nu}^{\rm tot} \equiv T_{\mu\nu} - \frac{\Lambda_c}{8\pi G}\, g_{\mu\nu}\,,$$

one can regain the original form of the field equations

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = 8\pi G T_{\mu\nu}^{tot} = 8\pi G \left(T_{\mu\nu} + T_{\mu\nu}^{\Lambda}\right), \tag{2}$$

at the prize of introducing a vacuum energy density and vacuum stress–energy tensor

$$\rho_\Lambda = \frac{\Lambda_c}{8\pi G}\,, \quad T_{\mu\nu}^{\Lambda} = -\rho_\Lambda g_{\mu\nu}\,.$$

Alternatively, Eq. (1) can be cast into the form,

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R + \Lambda_{\rm eff} g_{\mu\nu} = 0\,,$$

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where we have included the contribution of the vacuum energy density in the form $T_{\mu\nu} = -\langle\rho\rangle g_{\mu\nu}$. In this case Λ_c can be considered as the bare cosmological constant

$$\Lambda_{\text{eff}} = 8\pi G\rho_{\text{eff}} = \Lambda_c + 8\pi G\langle\rho\rangle\,.$$

Experimentally, we know that the effective energy density of the universe ρ_{eff} is of the order $10^{-47}\,\text{GeV}^4$. A crude estimate of the Zero Point Energy (ZPE) of some field of mass m with a cutoff at the Planck scale gives

$$E_{\text{ZPE}} = \frac{1}{2}\int_0^{\Lambda_p}\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2+m^2} \simeq \frac{\Lambda_p^4}{16\pi^2} \approx 10^{71}\,\text{GeV}^4\,. \tag{3}$$

This gives a difference of about 118 orders [1]. The approach to quantization of general relativity based on the following set of equations

$$\left[2\kappa G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\left(R - 2\Lambda_c\right)\right]\Psi\left[g_{ij}\right] = 0 \tag{4}$$

and

$$-2\nabla_i\pi^{ij}\Psi\left[g_{ij}\right] = 0\,, \tag{5}$$

where R is the three-scalar curvature, Λ_c is the bare cosmological constant and $\kappa = 8\pi G$, is known as Wheeler–De Witt equation (WDW) [2]. Eqs. (4) and (5) describe the *wave function of the universe*. The WDW equation represents invariance under *time* reparametrization in an operatorial form, while Eq. (5) represents invariance under diffeomorphism. G_{ijkl} is the *supermetric* defined as

$$G_{ijkl} = \frac{1}{2\sqrt{g}}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right).$$

Note that the WDW equation can be cast into the form

$$\left[2\kappa G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}R\right]\Psi\left[g_{ij}\right] = -\frac{\sqrt{g}}{\kappa}\Lambda_c\Psi\left[g_{ij}\right],$$

which formally looks like an eigenvalue equation. In this paper, we would like to use the Wheeler–De Witt (WDW) equation to estimate $\langle\rho\rangle$. In particular, we will compute the ZPE due to massive and massless gravitons propagating on the Schwarzschild background. This choice is dictated by considering that the Schwarzschild solution represents the only non-trivial static spherical symmetric solution of the Vacuum Einstein equations. Therefore, in this context the ZPE can be attributed only to quantum fluctuations. The used method will be a variational approach applied on gaussian wave functional. The rest of the paper is structured as follows, in section 2, we show how to apply the variational approach to the Wheeler–De Witt equation and we give some of the basic rules to solve such an equation approximated to second order in metric perturbation, in section 3, we analyze the spin-2 operator or the operator acting on transverse traceless tensors, in section 4 we use the zeta function to regularize the divergences coming from the evaluation of the ZPE for TT tensors and we write the renormalization group equation, in section 5 we use the same procedure of section 4, but for massive gravitons. We summarize and conclude in section 6.

2 The Wheeler−De Witt equation and the cosmological constant

The WDW equation (4), written as an eigenvalue equation, can be cast into the form

$$\hat{\Lambda}_\Sigma \Psi\left[g_{ij}\right] = -\Lambda'(x)\Psi\left[g_{ij}\right], \tag{6}$$

where

$$\begin{cases} \hat{\Lambda}_\Sigma = 2\kappa G_{ijkl}\pi^{ij}\pi^{kl} - \dfrac{\sqrt{g}}{2\kappa}R\,, \\ \Lambda' = \dfrac{\Lambda}{\kappa}\sqrt{g}\,. \end{cases}$$

We, now multiply Eq. (6) by $\Psi^*\left[g_{ij}\right]$ and we functionally integrate over the three spatial metric g_{ij}, then after an integration over the hypersurface Σ, one can formally re-write the WDW equation as

$$\frac{1}{V}\frac{\int \mathcal{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi\left[g_{ij}\right]}{\int \mathcal{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\Psi\left[g_{ij}\right]} = \frac{1}{V}\frac{\left\langle \Psi \left| \int_\Sigma d^3x\hat{\Lambda}_\Sigma \right| \Psi \right\rangle}{\left\langle \Psi | \Psi \right\rangle} = \Lambda'\,. \tag{7}$$

The formal eigenvalue equation is a simple manipulation of Eq. (4). However, we gain more information if we consider a separation of the spatial part of the metric into a background term, $\bar{g}_{ij}$, and a perturbation, h_{ij},

$$g_{ij} = \bar{g}_{ij} + h_{ij}\,.$$

Thus eq. (7) becomes

$$\frac{\left\langle \Psi \left| \int_\Sigma d^3x\left[\hat{\Lambda}_\Sigma^{(0)} + \hat{\Lambda}_\Sigma^{(1)} + \hat{\Lambda}_\Sigma^{(2)} + \ldots\right] \right| \Psi \right\rangle}{\left\langle \Psi | \Psi \right\rangle} = \Lambda'\Psi\left[g_{ij}\right], \tag{8}$$

where $\hat{\Lambda}_\Sigma^{(i)}$ represents the i^{th} order of perturbation in h_{ij}. By observing that the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}R^{(3)}$ up to quadratic order and we get

$$\begin{aligned} \int_\Sigma d^3x\sqrt{\bar{g}}\Big[& -\tfrac{1}{4}\,h\triangle h + \tfrac{1}{4}\,h^{li}\triangle h_{li} - \tfrac{1}{2}\,h^{ij}\nabla_l\nabla_i h^l_j + \\ & + \tfrac{1}{2}\,h\nabla_l\nabla_i h^{li} - \tfrac{1}{2}\,h^{ij}R_{ia}h^a_j + \tfrac{1}{2}\,hR_{ij}h^{ij} + \tfrac{1}{4}\,h\left(R^{(0)}\right)h\Big]\,, \end{aligned} \tag{9}$$

where h is the trace of h_{ij} and $R^{(0)}$ is the three dimensional scalar curvature. To explicitly make calculations, we need an orthogonal decomposition for both π_{ij} and h_{ij} to disentangle gauge modes from physical deformations. We define the inner product

$$\langle h, k\rangle := \int_\Sigma \sqrt{g}G^{ijkl}h_{ij}(x)k_{kl}(x)\,d^3x\,,$$

by means of the inverse WDW metric G_{ijkl}, to have a metric on the space of deformations, i.e. a quadratic form on the tangent space at h_{ij}, with

$$G^{ijkl} = \left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right).$$

The inverse metric is defined on cotangent space and it assumes the form

$$\langle p,q\rangle := \int_\Sigma \sqrt{g} G_{ijkl} p^{ij}(x) q^{kl}(x)\, \mathrm{d}^3x\,,$$

so that

$$G^{ijnm}G_{nmkl} = \tfrac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right).$$

Note that in this scheme the "inverse metric" is actually the WDW metric defined on phase space. The desired decomposition on the tangent space of 3-metric deformations [3–6] is:

$$h_{ij} = \tfrac{1}{3}\, h g_{ij} + (L\xi)_{ij} + h^{\perp}_{ij}\,, \tag{10}$$

where the operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \tfrac{2}{3}\, g_{ij}\,(\nabla\cdot\xi)\,. \tag{11}$$

Thus the inner product between three-geometries becomes

$$\langle h,h\rangle := \int_\Sigma \sqrt{g} G^{ijkl} h_{ij}(x) h_{kl}(x) \mathrm{d}^3x = \int_\Sigma \sqrt{g}\left[-\tfrac{2}{3}\, h^2 + (L\xi)^{ij}(L\xi)_{ij} + h^{ij\perp}h^{\perp}_{ij}\right]. \tag{12}$$

With the orthogonal decomposition in hand we can define the trial wave functional as

$$\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right] = \mathcal{N}\Psi\left[h^{\perp}_{ij}\left(\overrightarrow{x}\right)\right]\Psi\left[h^{\Vert}_{ij}\left(\overrightarrow{x}\right)\right]\Psi\left[h^{\mathrm{trace}}_{ij}\left(\overrightarrow{x}\right)\right], \tag{13}$$

where

$$\begin{aligned}
\Psi\left[h^{\perp}_{ij}\left(\overrightarrow{x}\right)\right] &= \exp\left\{-\tfrac{1}{4}\left\langle hK^{-1}h\right\rangle^{\perp}_{x,y}\right\},\\
\Psi\left[h^{\Vert}_{ij}\left(\overrightarrow{x}\right)\right] &= \exp\left\{-\tfrac{1}{4}\left\langle (L\xi)\,K^{-1}\,(L\xi)\right\rangle^{\Vert}_{x,y}\right\},\\
\Psi\left[h^{\mathrm{trace}}_{ij}\left(\overrightarrow{x}\right)\right] &= \exp\left\{-\tfrac{1}{4}\left\langle hK^{-1}h\right\rangle^{\mathrm{trace}}_{x,y}\right\}.
\end{aligned}$$

The symbol "$\perp$" denotes the transverse-traceless tensor (TT) (spin 2) of the perturbation, while the symbol "$\Vert$" denotes the longitudinal part (spin 1) of the perturbation. Finally, the symbol "trace" denotes the scalar part of the perturbation. $\mathcal{N}$ is a normalization factor, $\langle\cdot,\cdot\rangle_{x,y}$ denotes space integration and K^{-1} is the inverse "*propagator*". We will fix our attention to the TT tensor sector of the perturbation representing the graviton. Therefore, representation (13) reduces to

$$\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right] = \mathcal{N}\exp\left\{-\frac{1}{4}\left\langle hK^{-1}h\right\rangle^{\perp}_{x,y}\right\}. \tag{14}$$

Actually there is no reason to neglect longitudinal perturbations. However, following the analysis of Mazur and Mottola of Ref. [5] on the perturbation decomposition, we can discover that the relevant components can be restricted to the TT modes and to the trace modes. Moreover, for certain backgrounds TT tensors can be a source of instability as shown in Refs. [7]. Even the trace part can be regarded as a source of instability. Indeed this is usually termed *conformal* instability. The appearance of an instability on the TT modes is known as non conformal instability. This means that does not exist a gauge choice that can eliminate negative modes. To proceed with Eq. (8), we need to know the action of some basic operators on $\Psi\left[h_{ij}\right]$. The action of the operator h_{ij} on $|\Psi\rangle = \Psi\left[h_{ij}\right]$ is realized by [8]

$$h_{ij}(x)|\Psi\rangle = h_{ij}\left(\overrightarrow{x}\right)\Psi\left[h_{ij}\right].$$

The action of the operator π_{ij} on $|\Psi\rangle$, in general, is

$$\pi_{ij}(x)|\Psi\rangle = -\mathrm{i}\,\frac{\delta}{\delta h_{ij}\left(\overrightarrow{x}\right)}\,\Psi\left[h_{ij}\right],$$

while the inner product is defined by the functional integration:

$$\langle\Psi_1 \mid \Psi_2\rangle = \int\left[\mathcal{D}h_{ij}\right]\Psi_1^*\left[h_{ij}\right]\Psi_2\left[h_{kl}\right].$$

We demand that

$$\frac{1}{V}\,\frac{\left\langle\Psi\left|\int_\Sigma \mathrm{d}^3x\hat{\Lambda}_\Sigma\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle} = \frac{1}{V}\,\frac{\int\mathcal{D}\left[g_{ij}\right]\Psi^*\left[h_{ij}\right]\int_\Sigma \mathrm{d}^3x\hat{\Lambda}_\Sigma\Psi\left[h_{ij}\right]}{\int\mathcal{D}\left[g_{ij}\right]\Psi^*\left[h_{ij}\right]\Psi\left[h_{ij}\right]} \tag{15}$$

be stationary against arbitrary variations of $\Psi\left[h_{ij}\right]$. Note that Eq. (15) can be considered as the variational analog of a Sturm–Liouville problem with the cosmological constant regarded as the associated eigenvalue. Therefore the solution of Eq. (7) corresponds to the minimum of Eq. (15). The form of $\left\langle\Psi\left|\hat{\Lambda}_\Sigma\right|\Psi\right\rangle$ can be computed with the help of the wave functional (14) and with the help of

$$\begin{cases} \dfrac{\left\langle\Psi\left|h_{ij}\left(\overrightarrow{x}\right)\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle} = 0\,, \\ \dfrac{\left\langle\Psi\left|h_{ij}\left(\overrightarrow{x}\right)h_{kl}\left(\overrightarrow{y}\right)\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle} = K_{ijkl}\left(\overrightarrow{x},\overrightarrow{y}\right)\,. \end{cases}$$

Extracting the TT tensor contribution, we get

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V}\int_\Sigma \mathrm{d}^3x\sqrt{\bar{g}}G^{ijkl}\left[2\kappa K^{-1\perp}(x,x)_{ijkl} + \frac{1}{2\kappa}\left(\triangle_2\right)_j^a K^\perp(x,x)_{iakl}\right]. \tag{16}$$

The propagator $K^\perp(x,x)_{iakl}$ can be represented as

$$K^\perp\left(\overrightarrow{x},\overrightarrow{y}\right)_{iakl} := \sum_\tau \frac{h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right)h_{kl}^{(\tau)\perp}\left(\overrightarrow{y}\right)}{2\lambda(\tau)}\,, \tag{17}$$

where $h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right)$ are the eigenfunctions of $\triangle_2$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq. (16). The expectation value of $\hat{\Lambda}_{\Sigma}^{\perp}$ is easily obtained by inserting the form of the propagator into Eq. (16)

$$\Lambda'\left(\lambda_i\right) = \frac{1}{4}\sum_{\tau}\sum_{i=1}^{2}\left[2\kappa\lambda_i(\tau) + \frac{\omega_i^2(\tau)}{2\kappa\lambda_i(\tau)}\right].$$

By minimizing with respect to the variational function $\lambda_i(\tau)$, we obtain the total one loop energy density for TT tensors

$$\Lambda\left(\lambda_i\right) = -\frac{\kappa}{4}\sum_{\tau}\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right]. \tag{18}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$.

3 The transverse traceless (TT) spin 2 operator for the Schwarzschild metric and the W.K.B. approximation

The spin-two operator for the Schwarzschild metric is defined by

$$\left(\triangle_2 h^{\mathrm{TT}}\right)_i^j := -\left(\triangle_T h^{\mathrm{TT}}\right)_i^j + 2\left(R h^{\mathrm{TT}}\right)_i^j , \tag{19}$$

where the transverse-traceless (TT) tensor for the quantum fluctuation is obtained by the following decomposition

$$h_i^j = h_i^j - \tfrac{1}{3}\,\delta_i^j h + \tfrac{1}{3}\,\delta_i^j h = \left(h^T\right)_i^j + \tfrac{1}{3}\,\delta_i^j h\,.$$

This implies that $\left(h^T\right)_i^j \delta_j^i = 0$. The transversality condition is applied on $\left(h^T\right)_i^j$ and becomes $\nabla_j\left(h^T\right)_i^j = 0$. Thus

$$-\left(\triangle_T h^{\mathrm{TT}}\right)_i^j = -\triangle_S\left(h^{\mathrm{TT}}\right)_i^j + \frac{6}{r^2}\left(1 - \frac{2MG}{r}\right), \tag{20}$$

where $\triangle_S$ is the scalar curved Laplacian, whose form is

$$\triangle_S = \left(1 - \frac{2MG}{r}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{2r - 3MG}{r^2}\right)\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L^2}{r^2} \tag{21}$$

and R_j^a is the mixed Ricci tensor whose components are:

$$R_i^a = \left\{-\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3}\right\},$$

This implies that the scalar curvature is traceless. We are therefore led to study the following eigenvalue equation

$$\left(\triangle_2 h^{\mathrm{TT}}\right)_i^j = \omega^2 h_j^i\,, \tag{22}$$

where ω^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity [9]. In particular, our choice for the three-dimensional gravitational perturbation is represented by its even-parity form

$$\left(h^{\mathrm{even}}\right)^i_j(r,\vartheta,\phi)=\mathrm{diag}\left[H(r),K(r),L(r)\right]Y_{lm}(\vartheta,\phi)\,, \tag{23}$$

with

$$\begin{cases} H(r)=h^1_1(r)-\frac{1}{3}\,h(r)\,, \\ K(r)=h^2_2(r)-\frac{1}{3}\,h(r)\,, \\ L(r)=h^3_3(r)-\frac{1}{3}\,h(r)\,. \end{cases}$$

From the transversality condition we obtain $h^2_2(r)=h^3_3(r)$. Then $K(r)=L(r)$. For a generic value of the angular momentum L, representation (23) joined to Eq. (20) lead to the following system of PDE's

$$\begin{cases} \left(-\triangle_S+\dfrac{6}{r^2}\left(1-\dfrac{2MG}{r}\right)-\dfrac{4MG}{r^3}\right)H(r)=\omega^2_{1,l}H(r) \\ \left(-\triangle_S+\dfrac{6}{r^2}\left(1-\dfrac{2MG}{r}\right)+\dfrac{2MG}{r^3}\right)K(r)=\omega^2_{2,l}K(r)\,. \end{cases} \tag{24}$$

Defining reduced fields

$$H(r)=\frac{f_1(r)}{r}\qquad K(r)=\frac{f_2(r)}{r}$$

and passing to the proper geodesic distance from the *throat* of the bridge

$$\mathrm{d}x=\pm\frac{\mathrm{d}r}{\sqrt{1-\frac{2MG}{r}}}\,, \tag{25}$$

the system (24) becomes

$$\begin{cases} \left[-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}+V_1(r)\right]f_1(x)=\omega^2_{1,l}f_1(x)\,, \\ \left[-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}+V_2(r)\right]f_2(x)=\omega^2_{2,l}f_2(x) \end{cases} \tag{26}$$

with

$$\begin{cases} V_1(r)=\dfrac{l(l+1)}{r^2}+U_1(r)\,, \\ V_2(r)=\dfrac{l(l+1)}{r^2}+U_2(r)\,, \end{cases}$$

where we have defined $r\equiv r(x)$ and

$$\begin{cases} U_1(r)=\left[\dfrac{6}{r^2}\left(1-\dfrac{2MG}{r}\right)-\dfrac{3MG}{r^3}\right]\,, \\ U_2(r)=\left[\dfrac{6}{r^2}\left(1-\dfrac{2MG}{r}\right)+\dfrac{3MG}{r^3}\right]\,. \end{cases}$$

Note that

$$\begin{cases} U_1(r) \geq 0\,, & \text{when} \quad r \geq \frac{5}{2}\,MG\,, \\ U_1(r) < 0 & \text{when} \quad 2MG \leq r < \frac{5}{2}\,MG\,, \\ U_2(r) > 0 & \qquad\quad \forall\, r \in [2MG, +\infty)\ . \end{cases} \tag{27}$$

The functions $U_1(r)$ and $U_2(r)$ play the rôle of two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$, respectively. In order to use the WKB approximation, we define two r-dependent radial wave numbers $k_1\left(r, l, \omega_{1,nl}\right)$ and $k_2\left(r, l, \omega_{2,nl}\right)$

$$\begin{cases} k_1^2\left(r, l, \omega_{1,nl}\right) = \omega_{1,nl}^2 - \dfrac{l(l+1)}{r^2} - m_1^2(r)\,, \\ k_2^2\left(r, l, \omega_{2,nl}\right) = \omega_{2,nl}^2 - \dfrac{l(l+1)}{r^2} - m_2^2(r) \end{cases} \tag{28}$$

for $r \geq \frac{5}{2}\,MG$. When $2MG \leq r < \frac{5}{2}\,MG$, $k_1^2\left(r, l, \omega_{1,nl}\right)$ becomes

$$k_1^2\left(r, l, \omega_{1,nl}\right) = \omega_{1,nl}^2 - \frac{l(l+1)}{r^2} + m_1^2(r)\,. \tag{29}$$

4 One loop energy regularization and renormalization

In this section, we proceed to evaluate Eq. (18). The method is equivalent to the scattering phase shift method and to the same method used to compute the entropy in the brick wall model. We begin by counting the number of modes with frequency less than ω_i, $i = 1, 2$. This is given approximately by

$$\tilde{g}\left(\omega_i\right) = \sum_l \nu_i\left(l, \omega_i\right)(2l+1)\,, \tag{30}$$

where $\nu_i\left(l, \omega_i\right)$, $i = 1, 2$ is the number of nodes in the mode with $\left(l, \omega_i\right)$, such that $\big(r \equiv r(x)\big)$

$$\nu_i\left(l, \omega_i\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x \sqrt{k_i^2\left(r, l, \omega_i\right)}\,. \tag{31}$$

Here it is understood that the integration with respect to x and l is taken over those values which satisfy $k_i^2\left(r, l, \omega_i\right) \geq 0$, $i = 1, 2$. With the help of Eqs. (30, 31), we obtain the one loop total energy for TT tensors

$$\frac{1}{8\pi}\sum_{i=1}^{2}\int_{-\infty}^{+\infty} \mathrm{d}x \left[\int_0^{+\infty} \omega_i \, \frac{\mathrm{d}\tilde{g}(\omega_i)}{\mathrm{d}\omega_i}\,\mathrm{d}\omega_i\right].$$

By extracting the energy density contributing to the cosmological constant, we get

$$\begin{aligned} \Lambda &= \Lambda_1 + \Lambda_2 = \rho_1 + \rho_2 = \\ &= -\frac{\kappa}{16\pi^2}\int_0^{+\infty} \omega_1^2 \sqrt{\omega_1^2 - m_1^2(r)}\,\mathrm{d}\omega_1 - \frac{\kappa}{16\pi^2}\int_0^{+\infty} \omega_2^2 \sqrt{\omega_2^2 - m_2^2(r)}\,\mathrm{d}\omega_2\,, \end{aligned} \tag{32}$$

where we have included an additional 4π coming from the angular integration. We use the zeta function regularization method to compute the energy densities ρ_1 and ρ_2. Note that this procedure is completely equivalent to the subtraction procedure of the Casimir energy computation where the zero point energy (ZPE) in different backgrounds with the same asymptotic properties is involved. To this purpose, we introduce the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. Then we have

$$\rho_i(\varepsilon) = \frac{1}{16\pi^2}\,\mu^{2\varepsilon}\int_0^{+\infty} d\omega_i \frac{\omega_i^2}{\left(\omega_i^2 - m_i^2(r)\right)^{\varepsilon-1/2}}\,. \tag{33}$$

The integration has to be meant in the range where $\omega_i^2 - m_i^2(r) \geq 0$[1]. One gets

$$\rho_i(\varepsilon) = \kappa\frac{m_i^2(r)}{256\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m_i^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right], \tag{34}$$

$i = 1, 2$. In order to renormalize the divergent ZPE, we write

$$\Lambda = 8\pi G\big(\rho_1(\varepsilon) + \rho_2(\varepsilon) + \rho_1(\mu) + \rho_2(\mu)\big),$$

where we have separated the divergent part from the finite part. To handle with the divergent energy density we extract the divergent part of Λ, in the limit $\varepsilon \to 0$ and we set

$$\Lambda^{\mathrm{div}} = \frac{G}{32\pi\varepsilon}\left(m_1^4(r) + m_2^4(r)\right).$$

Thus, the renormalization is performed via the absorption of the divergent part into the re-definition of the bare classical constant Λ

$$\Lambda \to \Lambda_0 + \Lambda^{\mathrm{div}}\,.$$

The remaining finite value for the cosmological constant reads

$$\begin{aligned}\frac{\Lambda_0}{8\pi G} = \frac{1}{256\pi^2}\Bigg\{ & m_1^4(r)\left[\ln\left(\frac{\mu^2}{|m_1^2(r)|}\right) + 2\ln 2 - \frac{1}{2}\right] + \\ & + m_2^4(r)\left[\ln\left(\frac{\mu^2}{m_2^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right]\Bigg\} = \big(\rho_1(\mu) + \rho_2(\mu)\big) = \rho_{\mathrm{eff}}^{\mathrm{TT}}(\mu, r)\,.\end{aligned} \tag{35}$$

The quantity in Eq. (35) depends on the arbitrary mass scale μ. It is appropriate to use the renormalization group equation to eliminate such a dependence. To this aim, we impose that [10]

$$\frac{1}{8\pi G}\mu\,\frac{\partial\Lambda_0^{\mathrm{TT}}(\mu)}{\partial\mu} = \mu\frac{d}{d\mu}\,\rho_{\mathrm{eff}}^{\mathrm{TT}}(\mu, r)\,. \tag{36}$$

[1]) Details of the calculation can be found in the Appendix.

Solving it we find that the renormalized constant Λ_0 should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$\Lambda_0(\mu, r) = \Lambda_0(\mu_0, r) + \frac{G}{16\pi}\left(m_1^4(r) + m_2^4(r)\right)\ln\frac{\mu}{\mu_0}\,. \tag{37}$$

Substituting Eq. (37) into Eq. (35) we find

$$\begin{aligned}\frac{\Lambda_0(\mu_0, r)}{8\pi G} &= -\frac{1}{256\pi^2}\left\{m_1^4(r)\left[\ln\left(\frac{\left|m_1^2(r)\right|}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right] + \right.\\ &\left. + m_2^4(r)\left[\ln\left(\frac{m_2^2(r)}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right]\right\}.\end{aligned} \tag{38}$$

In order to fix the dependence of Λ on r and M, we find the minimum of $\Lambda_0(\mu_0, r)$. To this aim, last equation can be cast into the form[2])

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = -\frac{\mu_0^4}{256\pi^2}\left\{x^2(r)\left[\ln\left(\frac{|x(r)|}{4}\right) + \frac{1}{2}\right] + y^2(r)\left[\ln\left(\frac{y(r)}{4}\right) + \frac{1}{2}\right]\right\}, \tag{39}$$

where $x(r) = \pm m_1^2(r)/\mu_0^2$ and $y(r) = m_2^2(r)/\mu_0^2$. Now we find the extrema of $\Lambda_0\left(\mu_0; x(r), y(r)\right)$ in the range $\frac{5}{2}\,MG \leq r$ and we get

$$\begin{cases} x(r) = 0\,, \\ y(r) = 0\,, \end{cases} \tag{40}$$

which is never satisfied and

$$\begin{cases} x(r) = 4/\mathrm{e}\,, \\ y(r) = 4/\mathrm{e}\,, \end{cases} \Longrightarrow \begin{cases} m_1^2(r) = 4\mu_0^2/\mathrm{e}\,, \\ m_2^2(r) = 4\mu_0^2/\mathrm{e}\,, \end{cases} \tag{41}$$

which implies $M = 0$ and $\bar{r} = \sqrt{3\mathrm{e}}/2\mu_0$. On the other hand, in the range $2MG \leq r < \frac{5}{2}\,MG$, we get again

$$\begin{cases} x(r) = 0\,, \\ y(r) = 0\,, \end{cases}$$

which has no solution and

$$\begin{cases} -m_1^2(r) = 4\mu_0^2/\mathrm{e}\,, \\ m_2^2(r) = 4\mu_0^2/\mathrm{e}\,, \end{cases} \tag{42}$$

which implies

$$\begin{cases} \bar{M} = 4\mu_0^2\bar{r}^3/3\mathrm{e}G,\,, \\ \bar{r} = \sqrt{6\mathrm{e}}/4\mu_0\,. \end{cases} \tag{43}$$

[2]) Recall Eqs. (28,29), showing a change of sign in $m_1^2(r)$. Even if this is not the most appropriate notation to indicate a change of sign in a quantity looking like a "*square effective mass*", this reveals useful in the zeta function regularization and in the search for extrema.

Eq. (39) evaluated on the minimum, now becomes

$$\Lambda_0\left(\bar{M},\bar{r}\right)=\frac{\mu_0^4 G}{2\mathrm{e}^2\pi}\,,. \tag{44}$$

It is interesting to note that thanks to the renormalization group equation (36), we can directly compute Λ_0 at the scale μ_0 and only with the help of Eq. (37), we have access at the scale μ.

5 One loop energy regularization and renormalization for massive gravitons

The question of massive gravitons is quite delicate. A tentative to introduce a mass in the general framework has been done by Boulware and Deser [11], with the conclusion that the theory is unstable and produces ghosts. However, at the linearized level the Pauli–Fierz term [12]

$$S_{\text{P.F.}}=\frac{m_g^2}{8\kappa}\int \mathrm{d}^4x\sqrt{-g}\left[h^{\mu\nu}h_{\mu\nu}-h^2\right], \tag{45}$$

does not introduce ghosts. m_g is the graviton mass. Following Rubakov [13], the Pauli–Fierz term can be rewritten in such a way to explicitly violate Lorentz symmetry, but to preserve the three-dimensional Euclidean symmetry. In Minkowski space it takes the form

$$\begin{aligned}S_m=-\frac{1}{8\kappa}\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\big[&m_0^2h^{00}h_{00}+2m_1^2h^{0i}h_{0i}-m_2^2h^{ij}h_{ij}+\\&+m_3^2h^{ii}h_{jj}-2m_4^2h^{00}h_{ii}\big]\,.\end{aligned} \tag{46}$$

A comparison between the massive action (46) and the Pauli–Fierz term shows that they can be set equal if we make the following choice[3])

$$m_0^2=0\,,\quad m_1^2=m_2^2=m_3^2=m_4^2=m^2>0\,.$$

If we fix the attention on the case

$$m_0^2=m_1^2=m_3^2=m_4^2=0\,,\quad m_2^2=m^2>0\,, \tag{47}$$

we can see that the trace part disappears and we get

$$S_m=\frac{m_g^2}{8\kappa}\int \mathrm{d}^4x\sqrt{-\hat{g}}\left[h^{ij}h_{ij}\right].$$

The corresponding term in the linearized hamiltonian will be simply

$$\mathcal{H}_m=-\frac{m_g^2}{8\kappa}\int \mathrm{d}^3xN\sqrt{\hat{g}}\left[h^{ij}h_{ij}\right].$$

[3]) See also Dubovski [14] for a detailed discussion about the different choices of m_1, m_2, m_3 and m_4

This means that Eq. (19), will be modified into

$$\left(\triangle_2 h^{\mathrm{TT}}\right)_i^j := -\left(\triangle_T h^{\mathrm{TT}}\right)_i^j + 2\left(Rh^{\mathrm{TT}}\right)_i^j + \left(m_g^2 h^{\mathrm{TT}}\right)_i^j , \tag{48}$$

Therefore, the square effective mass will be modified by adding the term m_g^2. Note that, while $m_2^2(r)$ is constant in sign, $m_1^2(r)$ is not. Indeed, for the critical value $\bar{r} = 5MG/2$, $m_1^2(\bar{r}) = m_g^2$ and in the range $(2MG, 5MG/2)$ for some values of m_g^2, $m_1^2(\bar{r})$ can be negative. It is interesting therefore concentrate in this range. To further proceed, we observe that $m_1^2(r)$ and $m_2^2(r)$ can be recast into a more suggestive and useful form, namely

$$\begin{cases} m_1^2(r) = m_g^2 + U_1(r) = m_g^2 + m_1^2(r,M) - m_2^2(r,M)\,, \\ m_2^2(r) = m_g^2 + U_2(r) = m_g^2 + m_1^2(r,M) + m_2^2(r,M)\,, \end{cases}$$

where $m_1^2(r,M) \to 0$ when $r \to \infty$ or $r \to 2MG$ and $m_2^2(r,M) = 3MG/r^3$. Nevertheless, in the above mentioned range $m_1^2(r,M)$ is negligible when compared with $m_2^2(r,M)$. So, in a first approximation we can write

$$\begin{cases} m_1^2(r) \simeq m_g^2 - m_2^2(r_0,M) = m_g^2 - m_0^2(M)\,, \\ m_2^2(r) \simeq m_g^2 + m_2^2(r_0,M) = m_g^2 + m_0^2(M)\,, \end{cases}$$

where we have defined a parameter $r_0 > 2MG$ and $m_0^2(M) = 3MG/r_0^3$. The main reason for introducing a new parameter resides in the fluctuation of the horizon that forbids any kind of approach. Of course the quantum fluctuation must obey the uncertainty relations. Thus, the analogue of Eq. (38) for massive gravitons becomes

$$\begin{aligned} \frac{\Lambda_0(\mu_0,r)}{8\pi G} = -\frac{1}{256\pi^2}\Bigg\{ & \left(m_g^2 - m_0^2(M)\right)^2 \left[\ln\left(\frac{\left|m_g^2 - m_0^2(M)\right|}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right] + \\ & + \left(m_g^2 + m_0^2(M)\right)^2 \left[\ln\left(\frac{m_g^2 + m_0^2(M)}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right]\Bigg\}. \end{aligned} \tag{49}$$

We can now discuss three cases:

1. $m_g^2 \gg m_0^2(M)$,
2. $m_g^2 = m_0^2(M)$,
3. $m_g^2 \ll m_0^2(M)$.

In case 1), we can rearrange Eq. (49) to obtain

$$\begin{aligned}\frac{\Lambda_0(\mu_0,r)}{8\pi G} &= -\frac{1}{256\pi^2}\left\{2\left(m_g^4+m_0^4(M)\right)\left[\ln\left(\frac{m_g^2}{4\mu_0^2}\right)+\frac{1}{2}\right]+\right.\\ &\quad ++\left(m_g^4+m_0^4(M)\right)\left[\ln\left(1-\frac{m_0^2(M)}{m_g^2}\right)+\ln\left(1+\frac{m_0^2(M}{m_g^2}\right)\right]+\\ &\quad \left.+2m_g^2m_0^2(M)\left[\ln\left(1+\frac{m_0^2(M)}{m_g^2}\right)-\ln\left(1-\frac{m_0^2(M)}{m_g^2}\right)\right]\right\}\simeq\\ &\simeq -\frac{m_g^4}{256\pi^2}\left[2\ln\left(\frac{m_g^2}{4\mu_0^2}\right)+1+3\left(\frac{m_0^2(M)}{m_g^2}\right)^2\right].\end{aligned}$$

The last term can be rearranged to give

$$-\frac{m_g^4}{128\pi^2}\left[\ln\left(\frac{m_g^2}{4\mu_M^2}\right)+\frac{1}{2}\right],$$

where we have introduced an intermediate scale defined by

$$\mu_M^2=\mu_0^2\exp\left(-\frac{3m_0^4(M)}{2m_g^4}\right). \tag{50}$$

With the help of Eq. (50), the computation of the minimum of Λ_0^{TT} is more simple. Indeed, if

$$x=\frac{m_g^2}{4\mu_M^2},$$

Λ_0 becomes

$$\Lambda_{0,M}(\mu_0,x)=-\frac{G\mu_M^4}{\pi}x^2\left[\ln x+\frac{1}{2}\right]. \tag{51}$$

As a function of x, $\Lambda_{0,M}(\mu_0,x)$ vanishes for $x=0$ and $x=\mathrm{e}^{-1/2}$ and when $x\in\left[0,\exp\left(-\frac{1}{2}\right)\right]$, $\Lambda_{0,M}^{\mathrm{TT}}(\mu_0,x)\geq 0$. It has a maximum for

$$\bar{x}=\frac{1}{\mathrm{e}}\iff m_g^2=\frac{4\mu_M^2}{\mathrm{e}}=\frac{4\mu_0^2}{\mathrm{e}}\exp\left(-\frac{3m_0^4(M)}{2m_g^4}\right)$$

and its value is

$$\Lambda_{0,M}(\mu_0,\bar{x})=\frac{G\mu_M^4}{2\pi\mathrm{e}^2}=\frac{G\mu_0^4}{2\pi\mathrm{e}^2}\exp\left(-\frac{3m_0^4(M)}{m_g^4}\right)$$

or

$$\Lambda_{0,M}(\mu_0,\bar{x})=\frac{G}{32\pi}m_g^4\exp\left(\frac{3m_0^4(M)}{m_g^4}\right).$$

In case 2), Eq. (49) becomes

$$\frac{\Lambda_0(\mu_0,r)}{8\pi G}\simeq\frac{\Lambda_0(\mu_0)}{8\pi G}=-\frac{m_g^4}{128\pi^2}\left[\ln\left(\frac{m_g^2}{4\mu_0^2}\right)+\frac{1}{2}\right]$$

or

$$\frac{\Lambda_0(\mu_0)}{8\pi G}=-\frac{m_0^4(M)}{128\pi^2}\left[\ln\left(\frac{m_0^2(M)}{4\mu_0^2}\right)+\frac{1}{2}\right].$$

Again we define a dimensionless variable

$$x=\frac{m_g^2}{4\mu_0^2}$$

and we get

$$\frac{\Lambda_{0,0}(\mu_0,x)}{8\pi G}=-\frac{G\mu_0^4}{\pi}\,x^2\left[\ln x+\frac{1}{2}\right]. \tag{52}$$

The formal expression of Eq. (52) is very close to Eq. (51) and indeed the extrema are in the same position of the scale variable x, even if the meaning of the scale is here different. $\Lambda_{0,0}(\mu_0,x)$ vanishes for $x=0$ and $x=4\mathrm{e}^{-1/2}$. In this range, $\Lambda_{0,0}^{\mathrm{TT}}(\mu_0,x)\geq 0$ and it has a minimum located in

$$\bar{x}=\frac{1}{\mathrm{e}}\implies m_g^2=\frac{4\mu_0^2}{\mathrm{e}} \tag{53}$$

and

$$\Lambda_{0,0}(\mu_0,\bar{x})=\frac{G\mu_0^4}{2\pi\mathrm{e}^2}$$

or

$$\Lambda_{0,0}(\mu_0,\bar{x})=\frac{G}{32\pi}\,m_g^4=\frac{G}{32\pi}\,m_0^4(M)\,.$$

Finally the case 3) leads to

$$\frac{\Lambda_0(\mu_0,r)}{8\pi G}\simeq-\frac{m_0^4(M)}{256\pi^2}\left[2\ln\left(\frac{m_0^2(M)}{4\mu_0^2}\right)+1+3\left(\frac{m_g^2}{m_0^2(M)}\right)^2\right].$$

The last term can be rearranged to give

$$-\frac{m_0^4(M)}{128\pi^2}\left[\ln\left(\frac{m_0^2(M)}{4\mu_m^2}\right)+\frac{1}{2}\right],$$

where we have introduced another intermediate scale

$$\mu_m^2=\mu_0^2\exp\left(-\frac{3m_g^4}{2m_0^4(M)}\right).$$

By repeating the same procedure of previous cases, we define

$$x = \frac{m_0^2(M)}{4\mu_m^2}$$

and we get

$$\Lambda_{0,m}(\mu_0, x) = -\frac{G\mu_m^4}{\pi}\, x^2 \left[\ln x + \frac{1}{2}\right]. \tag{54}$$

Also this case has a maximum for

$$\bar{x} = \frac{1}{\mathrm{e}} \implies m_0^2(M) = \frac{4\mu_m^2}{\mathrm{e}} = \frac{4\mu_0^2}{\mathrm{e}} \exp\left(-\frac{3m_g^4}{2m_0^4(M)}\right).$$

and

$$\Lambda_{0,m}(\mu_0, \bar{x}) = \frac{G\mu_m^4}{2\pi \mathrm{e}^2} = \frac{G\mu_0^4}{2\pi \mathrm{e}^2} \exp\left(-\frac{3m_g^4}{m_0^4(M)}\right)$$

or

$$\Lambda_{0,M}(\mu_0, \bar{x}) = \frac{G}{32\pi}\, m_0^4(M) \exp\left(\frac{3m_g^4}{m_0^4(M)}\right).$$

Remark. Note that in any case, the maximum of Λ corresponds to the minimum of the energy density.

6 Summary and conclusions

In this paper, we have considered how to extract information on the cosmological constant using the Wheeler–De Witt equation when the graviton is massless and massive. In particular, by means of a variational approach and a orthogonal decomposition of the modes, we have studied the contribution of the transverse-traceless tensors in a Schwarzschild background. The use of the zeta function and a renormalization group equation have led to three different cases:

$$\begin{cases} m_g^2 \gg m_0^2(M) \\ m_g^2 = m_0^2(M) \\ m_g^2 \ll m_0^2(M) \end{cases} \implies \begin{cases} \Lambda_{0,M}(\mu_0, \bar{x}) = G\mu_0^4/\left(2\pi \mathrm{e}^2\right) \exp\left(-3m_0^4(M)/m_g^4\right) \\ \Lambda_{0,0}(\mu_0, \bar{x}) = G\mu_0^4/\left(2\pi \mathrm{e}^2\right) \\ \Lambda_{0,m}(\mu_0, \bar{x}) = G\mu_0^4/\left(2\pi \mathrm{e}^2\right) \exp\left(-3m_g^4/m_0^4(M)\right) \end{cases}$$

As we can see, the case "*extreme*", where the graviton mass is completely screened by the curvature "*mass*" seems to have the biggest value. We recall that the highest is the value of Λ_0 , the lowest is the value of the energy density. However, the expression of the extreme case coincides with the mass-less graviton discussed in section 4. In that paper, it is the curvature "*mass*" which plays the rôle of the mass of the graviton and contributes to the cosmological constant. So it appears that the gravitational field in the background of the Schwarzschild metric generates a "*mass*" term, because of the curvature and this term disappears when the Schwarzschild mass goes to zero. This leads to the conclusion that fluctuations around Minkowski space do not create a cosmological constant in absence of matter fields. Nevertheless, this behavior works if we accept that near the throat, vacuum fluctuations come into play forbidding to reach the throat itself. If this is not the case and the throat can be reached, then the curvature "*mass*" becomes completely non-perturbative when the Schwarzschild mass $M \to 0$. If we choose to fix the renormalization point $\mu_0 = m_p$, we obtain approximately $\Lambda_0^{\perp}(\bar{M}, \bar{r}) \simeq 10^{37}$ GeV^2 which, in terms of energy density is in agreement with the estimate of Eq. (3). Once fixed the scale μ_0, we can see what happens at the cosmological constant at the scale μ, by means of Eq. (37). What we see is that the cosmological constant is vanishing at the sub-planckian scale $\mu = m_p \mathrm{e}^{-1/4}$, but unfortunately is a scale which is very far from the nowadays observations. Note that, because of the condition (53), the graviton mass becomes proportional to the "Planck mass", which is of the order 10^{16} GeV, while the upper bound in eV is of the order $10^{-24} - 10^{-29}$ eV [15]. A quite curious thing comes on the estimate on the "square graviton mass", which in this context is closely related to the cosmological constant. Indeed, from Eq. (53) applied on the square mass, we get

$$m_g^2 \propto \mu_0^2 \simeq 10^{32}\ \mathrm{GeV}^2 = 10^{50}\ \mathrm{eV}^2\,,$$

while the experimental upper bound is of the order

$$\left(m_g^2\right)_{\mathrm{exp}} \propto 10^{-48} - 10^{-58}\ \mathrm{eV}^2\,,$$

which gives a difference of about $10^{98} - 10^{108}$ orders. This discrepancy strongly recall the difference of the cosmological constant estimated at the Planck scale with that measured in the space where we live. However, the analysis is not complete. Indeed, we have studied the spectrum in a W.K.B. approximation with the following condition $k_i^2(r, l, \omega_i) \geq 0$, $i = 1, 2$. Thus to complete the analysis, we need to consider the possible existence of nonconformal unstable modes, like the ones discovered in Refs. [7]. If such an instability appears, this does not mean that we have to reject the solution. In fact in Ref. [16], we have shown how to cure such a problem. In that context, a model of "space-time foam" has been introduced in a large N wormhole approach reproducing a correct decreasing of the cosmological constant and simultaneously a stabilization of the system under examination. Unfortunately in that approach a renormalization scheme was missing and a W.K.B. approximation on the wave function has been used to recover a Schrödinger-like equation. The possible next step is to repeat the scheme we have adopted here in a large N context, to recover the correct vanishing behavior of the cosmological constant.

A The zeta function regularization

In this appendix, we report details on computation leading to expression (33). We begin with the following integral

$$\rho(\varepsilon)=\begin{cases} I_+=\mu^{2\varepsilon}\displaystyle\int_0^{+\infty}\mathrm{d}\omega\frac{\omega^2}{\left(\omega^2+m^2(r)\right)^{\varepsilon-1/2}}\,, \\ I_-=\mu^{2\varepsilon}\displaystyle\int_0^{+\infty}\mathrm{d}\omega\frac{\omega^2}{\left(\omega^2-m^2(r)\right)^{\varepsilon-1/2}}\,, \end{cases} \tag{55}$$

with $m^2(r)>0$.

A.1 I_+ computation

If we define $t=\omega/\sqrt{m^2(r)}$, the integral I_+ in Eq. (55) becomes

$$\rho(\varepsilon)=\mu^{2\varepsilon}m^{4-2\varepsilon}(r)\int_0^{+\infty}\mathrm{d}t\frac{t^2}{\left(t^2+1\right)^{\varepsilon-1/2}}=\frac{1}{2}\,\mu^{2\varepsilon}m^{4-2\varepsilon}(r)B\left(\frac{3}{2},\varepsilon-2\right)\,,$$
$$\frac{1}{2}\,\mu^{2\varepsilon}m^{4-2\varepsilon}(r)\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(\varepsilon-2)}{\Gamma\left(\varepsilon-\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{4}\,m^4(r)\left(\frac{\mu^2}{m^2(r)}\right)^{\varepsilon}\frac{\Gamma(\varepsilon-2)}{\Gamma\left(\varepsilon-\frac{1}{2}\right)},$$

where we have used the following identities involving the beta function

$$B(x,y)=2\int_0^{+\infty}\mathrm{d}t\frac{t^{2x-1}}{\left(t^2+1\right)^{x+y}}\,,\quad \mathrm{Re}\,x>0\,,\quad \mathrm{Re}\,y>0$$

related to the gamma function by means of

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.$$

Taking into account the following relations for the Γ-function

$$\begin{aligned}\Gamma(\varepsilon-2)&=\frac{\Gamma(1+\varepsilon)}{\varepsilon(\varepsilon-1)(\varepsilon-2)}\,,\\ \Gamma(\varepsilon-\tfrac{1}{2})&=\frac{\Gamma(\varepsilon+\frac{1}{2})}{\varepsilon-\frac{1}{2}}\end{aligned} \tag{56}$$

and the expansion for small ε

$$\begin{aligned}\Gamma(1+\varepsilon)&=1-\gamma\varepsilon+O\left(\varepsilon^2\right),\\ \Gamma(\varepsilon+\tfrac{1}{2})&=\Gamma(\tfrac{1}{2})-\varepsilon\Gamma(\tfrac{1}{2})(\gamma+2\ln 2)+O\left(\varepsilon^2\right),\\ x^{\varepsilon}&=1+\varepsilon\ln x+O\left(\varepsilon^2\right),\end{aligned}$$

where γ is the Euler's constant, we find

$$\rho(\varepsilon)=-\frac{m^4(r)}{16}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2(r)}\right)+2\ln 2-\frac{1}{2}\right].$$

A.2 I_- computation

If we define $t = \omega/\sqrt{m^2(r)}$, the integral I_- in Eq. (55) becomes

$$\rho(\varepsilon) = \mu^{2\varepsilon} m^{4-2\varepsilon}(r) \int_0^{+\infty} \mathrm{d}t \frac{t^2}{(t^2-1)^{\varepsilon-1/2}} = \frac{1}{2}\,\mu^{2\varepsilon} m^{4-2\varepsilon}(r) B\left(\varepsilon-2, \frac{3}{2}-\varepsilon\right),$$

$$\frac{1}{2}\,\mu^{2\varepsilon} m^{4-2\varepsilon}(r) \frac{\Gamma(\frac{3}{2}-\varepsilon)\Gamma(\varepsilon-2)}{\Gamma(-\frac{1}{2})} = -\frac{1}{4\sqrt{\pi}}\, m^4(r) \left(\frac{\mu^2}{m^2(r)}\right)^{\varepsilon} \Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma(\varepsilon-2),$$

where we have used the following identity involving the beta function

$$\frac{1}{p}\, B\left(1-\nu-\frac{\mu}{p}, \nu\right) = \int_1^{+\infty} \mathrm{d}t\, t^{\mu-1}\left(t^p-1\right)^{\nu-1}$$
$$p > 0\,, \quad \mathrm{Re}\,\nu > 0\,, \quad \mathrm{Re}\,\mu < p - p\,\mathrm{Re}\,\nu$$

and the reflection formula

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z)\,.$$

From the first of Eqs. (56) and from the expansion for small ε

$$\Gamma(\tfrac{3}{2}-\varepsilon) = \Gamma(\tfrac{3}{2})\big(1-\varepsilon(-\gamma-2\ln 2+2)\big) + O\left(\varepsilon^2\right)$$
$$x^{\varepsilon} = 1 + \varepsilon \ln x + O\left(\varepsilon^2\right),$$

we find

$$\rho(\varepsilon) = -\frac{m^4(r)}{16}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right].$$

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