Deformed solitons: The case of two coupled scalar fields

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In this work, we present a general procedure, which is able to generate new exact solitonic models in 1+1 dimensions, from a known one, consisting of two coupled scalar fields. An interesting consequence of the method, is that of the appearing of nontrivial extensions, where the deformed systems presents other BPS solitons than that appearing in the original model. Finally we take a particular example, in order to check the above mentioned features.

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Despite of being less usual than linear systems, the nonlinear ones and particularly those having solitonic excitations are very interesting and important in order to modelling many physical, biological and chemical systems. A very important example is that of the electrical conductivity of some organic materials, where polarons and other polymer chain solitons were responsible for the appearing of conducting polymers [1]. Another important appearance of solitons is that related to electrical conduction through DNA molecules [2]. In the literature there exists a great number of models and applications of the solitonic solutions of one and two dimensional kind [3], [4].

In a recent paper, Bazeia et al [5] introduced a method capable of generating new exact solitonic systems from known ones, for the case of a scalar field in 1+1 dimensions. In this work, we intend to study the difficulties and restrictions of a naive generalization of the cited approach, and then present a general manner of getting new exact solitonic models from a known one, this time for the case of two or more coupled scalar fields.

The Lagrangian density for the the case of two coupled scalar fields which we are going to work with is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\left(\partial_\mu \chi\right)^2 - V\left(\phi, \chi\right), \tag{1}$$

whose Euler–Lagrange equations in 1+1 dimensions are given by

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = W_\phi W_{\phi\phi} + W_\chi W_{\chi\phi}\,, \quad \frac{\mathrm{d}^2\chi(x)}{\mathrm{d}x^2} = W_\chi W_{\chi\chi} + W_\phi W_{\phi\chi}\,, \tag{2}$$

where we particularized the potential to a class which can be written in terms of a superpotential W, as

$$V(\phi,\chi) = \frac{1}{2}\,W_\phi^2 + \frac{1}{2}\,W_\chi^2\,, \tag{3}$$

and W_ϕ and W_χ stands for, respectively, the differentiation with respect to the fields appearing in the lower index. For this class of systems, one can show that the minimum energy solutions can be obtained from the equivalent system of coupled first-order differential equations [7]

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = W_\phi(\phi,\chi)\,, \quad \frac{\mathrm{d}\chi}{\mathrm{d}x} = W_\chi(\gamma,\chi)\,. \tag{4}$$

If one starts from the above differential equations, and recover the corresponding second-order ones, one gets

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = W_\phi W_{\phi\phi} + W_\chi W_{\phi\chi}\,, \quad \frac{\mathrm{d}^2\chi(x)}{\mathrm{d}x^2} = W_\chi W_{\chi\chi} + W_\phi W_{\chi\phi}\,, \tag{5}$$

which are identical to those coming from the Euler–Lagrange equation as written above, provided that the superpotential be twicely differentiable. In other words, the following restriction shows up

$$W_{\phi\chi} = W_{\chi\phi}\,. \tag{6}$$

The energy of the so called BPS states can be calculated straightfowardly, giving

$$E_B = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\left[\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}\chi}{\mathrm{d}x}\right)^2 + \frac{1}{2}\,W_\phi^2 + \frac{1}{2}\,W_\chi^2\right], \tag{7}$$

which lead us to

$$E_B = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\left[\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} - W_\phi\right)^2 + \left(\frac{\mathrm{d}\chi}{\mathrm{d}x} - W_\chi\right)^2 + W_\chi\frac{\mathrm{d}\chi}{\mathrm{d}x} + W_\phi\frac{\mathrm{d}\phi}{\mathrm{d}x}\right], \tag{8}$$

from which we can see that the minimal energy will come from the solutions obeying the following set of first-order differential equations

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = W_\phi\,; \quad \frac{\mathrm{d}\chi}{\mathrm{d}x} + W_\chi \tag{9}$$

and the energy of the field configuration is finally given by

$$E_B = \left|W\left(\phi_i,\chi_i\right) - W\left(\phi_j,\chi_j\right)\right|, \tag{10}$$

where ϕ_i and χ_i are the i–th vacuum state of the model [9].

At this point we perform a general transformation in the fields

$$\phi = f(\theta,\varphi)\,, \quad \chi = g(\theta,\varphi)\,, \tag{11}$$

which after some simple manipulations lead us to

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = W_\theta(\theta,\varphi)\,, \quad \frac{\mathrm{d}\chi}{\mathrm{d}x} = W_\varphi(\theta,\varphi)\,, \tag{12}$$

where we defined

$$W_\theta(\theta,\varphi) = \frac{1}{J}\left(\frac{\partial g}{\partial\theta}\,W_\phi(\theta,\varphi) - \frac{\partial f}{\partial\theta}\,W_\chi(\theta,\varphi)\right), \tag{13}$$

and

$$W_\varphi(\theta,\varphi) = \frac{1}{J}\left(\frac{\partial f}{\partial \varphi}\, W_\chi(\theta,\varphi) - \frac{\partial g}{\partial \varphi}\, W_\chi(\theta,\varphi)\right), \tag{14}$$

with the Jacobian of the transformation given as usually by

$$J(\theta,\varphi) = \frac{\partial g}{\partial \theta}\,\frac{\partial f}{\partial \varphi} - \frac{\partial f}{\partial \theta}\,\frac{\partial g}{\partial \varphi}\,. \tag{15}$$

Unfortunately however, the derivative of the superpotential appearing at the right-hand side of the equations (12) does not obeys the rule appearing in (6), which in its turns is strictly necessary to guarantee that the solutions of the first-order equations are also solutions of the corresponding second-order ones, as must happens when studying the so called BPS solitons [8]. In order to become clearer the situation, we exemplify the idea by studying a particular example inspired in one proposed in the paper by Bazeia et al [5]. Namely we have

$$\phi = \sinh(\varphi)\,, \quad \chi = \theta\,. \tag{16}$$

Furthemore we apply it to a well known model presenting solitonic solutions [7],

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \lambda\left(\phi^2 - a^2\right) + \mu\chi^2\,, \quad \frac{\mathrm{d}\chi}{\mathrm{d}x} = -2\mu\phi\chi\,, \tag{17}$$

which, after performing the necessary calculations introduced above, leaves us with the following set of equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \mathrm{sech}(\varphi)\left[\lambda\left(\mathrm{senh}(\varphi)^2 - a^2\right) + \mu\theta^2\right], \quad \frac{\mathrm{d}\theta}{\mathrm{d}x} = -2\mu\theta\,\mathrm{senh}(\varphi)\,. \tag{18}$$

It is easy to verify that, indeed, by using solutions of the system (17) as

$$\phi = -a\tanh(2\mu a x)\,, \quad \chi = \pm\, a\sqrt{\frac{\lambda}{\mu} - 2}\;\mathrm{sech}(2\mu a x)\,, \tag{19}$$

one obtains the correct solution of the system of equations (18),

$$\varphi(x) = \mathrm{argsinh}\big(-a\tanh(2\mu a x)\big)\,, \quad \theta(x) = \pm\, a\sqrt{\frac{\lambda}{\mu} - 2}\;\mathrm{sech}(2\mu a x)\,. \tag{20}$$

Notwithstanding, the above solutions of the first-order differential equations are not solutions for the corresponding second-order ones. This happens precisely due to the fact that $W_{\theta\varphi} \neq W_{\varphi\theta}$. From now on, we are going to present an approach which is able to recover two new deformed nonlinear systems, from the above ones, which accomplish with the conditions to have BPS states. For reach this goal, we start by noting that the superpotential can be determined from each one of the equations (12), giving

$$W^{(1)}(\theta,\varphi) = \int \mathcal{D}\theta\, W_\theta(\theta,\varphi) + H^{(1)}(\varphi)\,, \tag{21}$$

or alternatively

$$W^{(2)}(\theta,\varphi)=\int\mathcal{D}\varphi\, W_{\varphi}(\theta,\varphi)+H^{(2)}(\theta)\,, \tag{22}$$

where $H^{(1)}(\varphi)$ and $H^{(2)}(\theta)$ are arbitrary functions which will be fixed in order to guarantee that the condition $W_{\theta\varphi}^{(i)}=W_{\varphi\theta}^{(i)}$, $(i=1,2)$ be satisfied. Now, imposing that one of the solutions described in (21) or (22) satisfies the condition (6), we obtain respectively

$$W_{\theta\varphi}(\theta,\varphi)+H_{\varphi}^{(1)}(\varphi)=W_{\varphi\theta}(\theta,\varphi)\,, \tag{23}$$

and

$$W_{\varphi\theta}(\theta,\varphi)+H_{\theta}^{(2)}(\theta)=W_{\theta\varphi}(\theta,\varphi)\,. \tag{24}$$

The last step is to determine the arbitrary function $H_{\varphi}^{(1)}(\varphi)$ or $H_{\theta}^{(2)}(\theta)$, by using our knowlegment of the relation between the original and the transformed fields, obtainable from the invertion of the transformations (11),

$$\theta=f(\phi,\chi)\,,\quad \varphi=g(\phi,\chi)\,, \tag{25}$$

and also of the solutions of those original fields $\phi(x)$ and $\chi(x)$. Then it is possible to write one field in terms of the another one ($\theta=\theta(\varphi)$ or $\varphi=\varphi(\theta)$). So, one can finally discover the expression of $H^{(1)}(\varphi)$ or $H^{(2)}(\theta)$. In doing so, one can recover two systems of BPS equations:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=W_{\theta}(\theta,\varphi)\,,\quad \frac{\mathrm{d}\varphi}{\mathrm{d}x}=W_{\varphi}(\theta,\varphi)+H_{\varphi}^{(1)}(\varphi)\,, \tag{26}$$

or

$$\frac{\mathrm{d}\chi}{\mathrm{d}x}=W_{\varphi}(\theta,\varphi)\,,\quad \frac{\mathrm{d}\theta}{\mathrm{d}x}=W_{\theta}(\theta,\varphi)+H_{\theta}^{(2)}(\theta)\,. \tag{27}$$

Let us now present a concrete realization of the idea above presented. We start by treating the case discussed here when we was showing that a naive generalization of the idea outlined in [5] does not works for the construction of deformed solitons when two or more coupled fields are present. Using the solutions (20), it is easy to verify that

$$\theta^2(x)=\left(\frac{\lambda}{\mu}-2\right)\left(a^2-\sinh\varphi^2\right), \tag{28}$$

and now for instance imposing the requirement (23), one obtains after straightforward calculations that

$$H_{\varphi}^{(1)}(\varphi)=\left(\sinh\varphi^2-a^2\right)\left[2\mu\,\mathrm{sech}\,\varphi+(\lambda-2\mu)\cosh\varphi\right], \tag{29}$$

whose corresponding set of coupled BPS equations are

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=-2\mu\theta\,\mathrm{senh}(\varphi)\,, \tag{30}$$

and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=\mu\cosh(\varphi)\,\theta^2+\left(\mathrm{senh}(\varphi)^2-a^2\right)\left[2\mu\,\mathrm{sech}\,\varphi+(\lambda-2\mu)\cosh\varphi\right]. \tag{31}$$

It is easy to verify now that this last system has the correct behavior as a BPS one [8]. In other words, they generate a potential of the type appearing in (3), coming from a superpotential given by

$$W_1 = \mu \sinh(\varphi)\, \theta^2 + H_1(\varphi)\,, \tag{32}$$

where

$$H_1(\varphi) = \int \mathrm{d}\varphi \left(\sinh \varphi^2 - a^2\right) \left[2\mu \,\mathrm{sech}\, \varphi + (\lambda - 2\mu) \cosh \varphi\right]. \tag{33}$$

Concluding the work, we should say that we are working on the extension of this approach to the case of non-BPS states, the consequences for the appearing of bags, junctions and networks of topological defects [9], a more extensive exploration of the new models coming from the approach proposed here, quantum extensions of this classical method and to the case of a greater number of fields [10].

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