Chaining spins from (super)Yang−Mills

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We review the spin bit model describing anomalous dimensions of the operators of Super Yang–Mills theory. We concentrate here on the scalar sector. In the limit of large N this model coincides with integrable spin chain while at finite N it has nontrivial chain splitting and joining interaction.

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1 Introduction

Effective description of gauge interactions in terms of string has a long history [1] (see [2] for a more recent review). Last years the development of the field has streamlined into what is know as AdS/CFT (or Maldacena) conjecture [3, 4], which claims that superstring theory in the background of $\mathrm{AdS}_5 \times S^5$ is essentially the same as $\mathcal{N} = 4$ Super Yang–Mills (SYM) model in four–dimensional Minkowski space which is the topological boundary of AdS_5. The assumption is based on the fact that two theories have the same symmetry group whose bosonic part $SO(2,4) \times SU(4)$ is one one hand the symmetry group of $\mathrm{AdS}_5 \times S^5$ space and one the other hand is the conformal group of four–dimensional Minkowski space. (Due to vanishing of β–functions $\mathcal{N} = 4$ SYM model is a conformally invariant model.)

Identifying irreducible representation of both groups one can put into one-to-one correspondence operators in SYM and states in the string theory. In particular, anomalous dimensions of operators in SYM theory (which are eigenvalues of the dilations) correspond to energy levels in string theory (see [5] for a review of AdS/CFT correspondence).

Recently much progress was achieved in understanding the scaling properties of SYM operators (see e.g. [6]). In particular, it was found for the scalar sector of SYM theory operators that in the planar limit the anomalous dimension matrix can be mapped into the Hamiltonian of integrable $SU(4) \sim SO(6)$ spin chain [7], this result was further generalized to the whole symmetry supergroup $SU(2,2|4)$ in [8]. Thus, Bethe Ansatz allows one to find the planar anomalous dimension of any operator in SYM without computing explicitly the corresponding Feynman diagrams. This spin chain is supposed to be a discrete version of the string in AdS background.

Going beyond the planar limit results in allowing chains to split and join. (When a fixed number of impurities is considered this dynamics can be described in terms of a quantum mechanical system like one in [9].) The natural task is, then, to extend the spin chain description to the nonplanarity. Indeed, one can do a one-to-

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one map of SYM operators into a system of interacting spins — *spin bits* such that the anomalous dimension matrix of SYM operators maps to a Hamiltonian for this spin system at finite N. In this note we are going to review this model. Interesting reader can find more details in the original paper [10].

2 Preliminaries

We consider scalar k–trace operators of the type[1]),

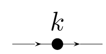
$$\mathcal{O} = \mathrm{tr}\, \phi_{i_1^{(1)}} \cdots \phi_{i_{L_1}^{(1)}}\, \mathrm{tr}\, \phi_{i_1^{(2)}} \cdots \phi_{i_{L_1}^{(2)}} \ldots \mathrm{tr}\, \phi_{i_1^{(k)}} \cdots \phi_{i_{L_k}^{(k)}}\ , \tag{1}$$

where L_i are the lengths of traces and $L = L_1 + L_2 + \cdots + L_k$ is the total length. The above operators can be equivalently represented in the following form,

$$\mathcal{O} \equiv |s;\gamma\rangle = \phi_{i_1}^{a_1 a_{\gamma(1)}} \phi_{i_2}^{a_2 a_{\gamma(2)}} \ldots \phi_{i_L}^{a_L a_{\gamma(L)}}, \tag{2}$$

where γ is an element of the permutation group of $\{1, 2, \ldots, L\}$ and s labels the indices $i_1, i_2, \ldots, i_L$. In particular, the operator (1) corresponds to a permutation with cycles $(L_1)(L_2)\ldots(L_k)$. It is not difficult to see that to each permutation one can put into correspondence a trace of operators if one specifies the index of each letter.

Graphically the field ϕ insertion will be represented by a site with two valent lines, like follows:



The incoming arrow connects with the site $\gamma^{-1}(k)$ while the outgoing one goes to $\gamma(k)$. In general, the arrow denote the action of the permutation.

3 The Hamiltonian

The Hamiltonian in the combinatorial form is obtained by computing the action of the operator [11]

$$H \equiv \Delta_{(2)} = -: \mathrm{tr}[\phi_m, \phi_n][\check{\phi}_m, \check{\phi}_n] : -\tfrac{1}{2} : \mathrm{tr}[\phi_m, \check{\phi}_n][\phi_m, \check{\phi}_n] :, \tag{3}$$

where,

$$\check{\phi}_i^{ab} = \frac{\partial}{\partial \phi_i^{ba}}\ , \tag{4}$$

and colons denote that derivatives do not act on other fields in the group.

Let us compute directly the action of the operator (3) on a state $|s;\gamma\rangle$,

$$\begin{aligned} H\,|s;\gamma\rangle = \Big[& 2(\delta^{j_1 j_4}\delta^{j_2 j_3} - \delta^{j_1 j_3}\delta^{j_2 j_4}) : \mathrm{tr}\, \phi_{j_1}\phi_{j_2}\check{\phi}_{j_3}\check{\phi}_{j_4} : + \\ & + \delta^{j_1 j_3}\delta^{j_2 j_4} : \mathrm{tr}\, \phi_{j_1}\check{\phi}_{j_2}\phi_{j_3}\check{\phi}_{j_4} : + \\ & + \delta^{j_1 j_4}\delta^{j_2 j_3} : \mathrm{tr}\, \phi_{j_1}\check{\phi}_{j_2}\check{\phi}_{j_3}\phi_{j_4} : \Big] \big(\phi_{i_1}^{a_1 a_{\gamma(1)}} \ldots \phi_{i_L}^{a_L a_{\gamma(L)}}\big). \end{aligned} \tag{5}$$

[1]) The number of traces can vary, it is a dynamical quantity

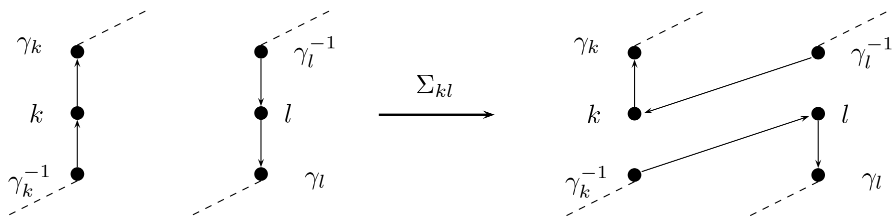


Fig. 1. Action of Σ_{kl} on the permutation state γ.

Application of the operator from the second line of (5) yields,

$$\begin{aligned}&2\delta^{a_k a_{\gamma(l)}}\left(\phi_{i_1}^{a_1 a_{\gamma(1)}}\dots\phi_{i_k}^{a a_{\gamma(k)}}\dots\phi_{i_l}^{a_l a}\dots\phi_{i_L}^{a_L a_{\gamma(L)}}\right)-\\&\qquad-2\delta^{a_k a_{\gamma(l)}}\left(\phi_{i_1}^{a_1 a_{\gamma(1)}}\dots\phi_{i_k}^{a_l a}\dots\phi_{i_l}^{a a_{\gamma(k)}}\dots\phi_{i_L}^{a_L a_{\gamma(L)}}\right).\end{aligned} \tag{6}$$

This corresponds to graphs with modified cyclic structure: $\gamma\mapsto\gamma\cdot\sigma_{k\gamma_l}$ and $\gamma\mapsto\sigma_{kl}\cdot\sigma_{k\gamma_l}\cdot\gamma\cdot\sigma_{kl}$ respectively for the first and second line of (6) (see fig. 1. Therefore, action of this part of the Hamiltonian (3) can be represented as,

$$2\big[|s;\gamma\cdot\sigma_{k\gamma_l}\rangle-|s;\sigma_{kl}\cdot\gamma\cdot\sigma_{k\gamma_l}\cdot\sigma_{kl}\rangle\big]. \tag{7}$$

Analogously, the second line of (5) produce the modified cycles given by,

$$\sum_{kl}K_{kl}\big[|s;\gamma\cdot\sigma_{k\gamma_l}\rangle-|s;\gamma\cdot\sigma_{\gamma_k\gamma_l}\rangle\big]. \tag{8}$$

Combining both results (7) and (8) together, one gets for the total Hamiltonian :

$$\begin{aligned}H|s;\gamma\rangle&=\sum_{kl}2\big(|s;\gamma\cdot\sigma_{k\gamma_l}\rangle-|s;\sigma_{kl}\cdot\gamma\cdot\sigma_{k\gamma_l}\cdot\sigma_{kl}\rangle\big)+\\&\quad+\sum_{kl}K_{kl}\big(|s\gamma\cdot\sigma_{k\gamma_l}\rangle-|s;\gamma\cdot\sigma_{\gamma_k\gamma_l}\rangle\big)=\\&=\sum_{kl}\big[2(1-P_{k\gamma^{-1}(l)})+(K_{k\gamma^{-1}(l)}-K_{\gamma^{-1}(k)\gamma^{-1}(l)})\big]|s\gamma\cdot\sigma_{kl}\rangle\equiv\\&\equiv\sum_{kl}(H_{k\gamma^{-1}(l)}-H_{\gamma^{-1}(k)\gamma^{-1}(l)})\Sigma_{kl}|s;\gamma\rangle.\end{aligned} \tag{9}$$

Here we introduced the joining/splitting operator Σ_{kl}, which change the permutation group element as follows,

$$\Sigma_{kl}|s;\gamma\rangle=\begin{cases}|s;\gamma\cdot\sigma_{kl}\rangle, & k\neq l,\\ N|s;\gamma\rangle, & k=l.\end{cases} \tag{10}$$

Factor N for coinciding k and l appears due to the fact that splitting and joining the trace in the same point results in multiplying by $\operatorname{tr}\mathrm{Id}=N$.

Note that since γ_k appears as index in the spin part (9) the order between Γ_{kl} and H_{kl} is important. One can, however re-sum (9) in order to get a form in which spin and permutation parts are completely independent and commute:

$$H = \sum_{k,l} H_{kl} (\Sigma_{k\gamma_l} - \Sigma_{\gamma_k\gamma_l}) \,. \tag{11}$$

4 The Hilbert space

Consider the description of the operators (2) in terms of the spin states. The L-field operator like (1) should be labelled by the spin variables $\{s_n^\alpha\}$ and a permutation group element γ.

The spin part itself is represented by a vector in the tensor product of one spin representations corresponding to each "site" n,

$$|s_1\rangle \otimes |s_2\rangle \cdots \otimes |s_L\rangle \equiv |s\rangle \,. \tag{12}$$

Obviously, one can choose an orthonormal, basis in the one–spin space such that, $s_n = s_n^\alpha e_\alpha$ and

$$\langle e_\alpha \mid e_\beta \rangle = \delta_{\alpha\beta} \,. \tag{13}$$

The "graph" part of the state we denote by $|\gamma\rangle$, $\gamma \in \Gamma_L$, permutation group of L elements. For the space of permutation there is a natural scalar product,

$$\langle \gamma \mid \gamma' \rangle = \delta_{\gamma\gamma'} \,, \tag{14}$$

i.e. it is zero for different permutations and unity when contracted with itself.

The states in our model should be represented as elements tensor product of the above two spaces modulo the symmetry group S_L,

$$|\{s\}, \gamma\rangle \in \{|\{s\}\rangle \otimes |\gamma\rangle\}/S_L \,, \tag{15}$$

where the symmetry group acts as follows,

$$\hat{\Sigma}_\sigma \, |s\rangle \otimes |\gamma\rangle = |s_\sigma\rangle \otimes \left|\sigma^{-1} \cdot \gamma \cdot \sigma\right\rangle , \quad \sigma \in S_L \,, \tag{16}$$

where

$$s_\sigma = \{s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(L)}\} \tag{17}$$

is a permutation of indices given by $\sigma \in S_L$. Indeed, as it is not very difficult to see the original and permuted states describe, in fact, the same trace in SYM model.

Given an arbitrary basis element $|s\rangle \otimes |\gamma\rangle$ one can find an element of the factor (15) by averaging with respect to the action of the group Γ_L,

$$|s; \gamma\rangle = \frac{1}{|S_L|} \sum_{\sigma \in S_L} |s_\sigma\rangle \otimes \left|\sigma^{-1} \cdot \gamma \cdot \sigma\right\rangle \equiv \hat{\Pi} \, |s\rangle \otimes |\gamma\rangle \,, \tag{18}$$

where Π is the cyclic symmetry projector,

$$\Pi = \frac{1}{|S_L|} \sum_{\sigma \in S_L} U_\sigma \otimes \hat{\Sigma}_\sigma \,, \tag{19}$$

where U_σ and $\hat{\Sigma}_\sigma$ are the following operators

$$U_\sigma = P_{1\sigma(1)} \otimes P_{2\sigma(2)} \cdots \otimes P_{L\sigma(L)} \,, \quad \hat{\Sigma}_\sigma \, |\gamma\rangle = \left|\sigma^{-1} \cdot \gamma \cdot \sigma\right\rangle \tag{20}$$

and $|S_L|$ stays for the order of S_L. As we mentioned Π is a projector, i.e. $\Pi^2 = \Pi$, also it is not difficult to see that Π commutes with permutation invariant operators.

Obviously, the above states are invariant with respect to the action of the gauge group S_L. In particular, when $\sigma = \gamma$ this symmetry represents the cyclicity of the trace(s).

5 Gauge symmetry

Let us show that the spin bit Hamiltonian (11) can be seen as arising from gauging of the planar spin chain. Since γ has the natural interpretation of the connection then the γ preserving symmetry $n \mapsto \sigma(n)$ has the meaning of the "global" gauge symmetry. (In fact this is translation symmetry.)

As we have seen in the previous section, an arbitrary permutation leads to more general transformation rules for "points" and "connections"

$$n \mapsto \sigma(n) \,, \quad \gamma \mapsto \sigma^{-1} \cdot \gamma \cdot \sigma \,. \tag{21}$$

Now, this is the localized version of the shift transformation, i.e. the discrete analog of the diffeomorphism transformations.

By direct evaluation of the Hamiltonian (11) one can show that it is invariant with respect to transformations (21). The Hamiltonian (11) can be rewritten in the following form,

$$H = \sum_{kl} V_{kl} H_{kl} \,, \tag{22}$$

where,

$$V_{kl} = \Sigma_{k\gamma_l} - \Sigma_{\gamma_k \gamma_l} \,. \tag{23}$$

In above expression V_{kl} can be expressed as the discrete gauge connection between sites k and l.

On the other hand, consider the planar spin chain Hamiltonian,

$$H_0 = \sum_k H_{k,k+1} = \sum_{kl} V^{(0)}_{kl} H_{kl} \,, \tag{24}$$

where,

$$V^{(0)}_{kl} = \delta_{k\gamma_0(l)} - \delta_{\gamma_0(k)\gamma_0(l)} \,, \quad \gamma_0(n) = n + 1 \,. \tag{25}$$

The above expression (25) differs from the complete nonplanar V_{kl} in eq. (23) by the only fact that operators Σ_{kl} are replaced by Kronecker delta symbols δ_{kl}.

Now it is clear, that passing from planar to general non-planar description amounts in "switching on" the gauge field operator V_{kl} (or Σ_{kl}), which plays the role of the gauge field. This procedure is sensitive to transformation properties of the fields to the permutation group action only and not on the structure of H_{kl}. Therefore, this procedure can be applied for obtaining the nonplanar Hamiltonian in the case of the spin chain with arbitrary group just passing from $V^{(0)}$ to V.

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References

[1] G. 't Hooft: Nucl. Phys. B **72** (1974) 461.

[2] G. 't Hooft: `hep-th/0204069`.

[3] J.M. Maldacena: Adv. Theor. Math. Phys. **2** (1998) 231; `hep-th/9711200`.

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov: Phys. Lett. B **428** (1998) 105; `hep-th/9802109`.

[5] O. Aharony, S. S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz: Phys. Rept. **323** (2000) 183; `hep-th/9905111`.

[6] J.C. Plefka: Fortsch. Phys. **52** (2004) 264; `hep-th/0307101`.

[7] J.A. Minahan and K. Zarembo: JHEP **03** (2003) 013; `hep-th/0212208`.

[8] N. Beisert and M. Staudacher: `hep-th/0307042`.

[9] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, Phys. Lett. B **558**, 229 (2003); `arXiv:hep-th/0212269`.

[10] S. Bellucci, P.Y. Casteill, J.F. Morales and C. Sochichiu: *Spin bit models from non-planar $N = 4$ SYM*, To Appear in: Nucl. Phys. B (2004); `hep-th/0404066`.

[11] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher: Nucl. Phys. B **650** (2003) 125; `hep-th/0208178`.