# Surfaces in $su(N)$ algebra via $\mathbb{C}P^{N-1}$ sigma models on Minkowski space

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We review our recent results concerning two–dimensional smooth orientable surfaces immersed in $su(N)$ Lie algebras. These are derived from the $\mathbb{C}P^{N-1}$ sigma model defined on Minkowski space. The structural equations of such surfaces expressed in terms of any regular solution of the $\mathbb{C}P^{N-1}$ model are found. This is carried out using a moving frame adapted to the surface. A procedure for construction of such surfaces is proposed and illustrated by several examples obtained from the $\mathbb{C}P^1$ model.

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## 1 Introduction

Recently problems related to surfaces immersed in $\mathbb{R}^n$ in connection with integrable systems have been researched extensively (for a review see [1] and the references therein). The progress in the analytic treatment of surfaces obtained from nonlinear differential equations has been rapid and resulted in many new techniques and theoretical approaches. Some of the most interesting developments have been in the study of surfaces immersed in Lie algebras, using techniques of completely integrable systems [2, 3, 4, 5, 6]. These surfaces are characterized by fundamental forms whose coefficients satisfy Gauss–Weingarten and Gauss–Codazzi–Ricci equations.

In this paper we review our results obtained in [7] by application of group–theoretical approach to surfaces associated to the $\mathbb{C}P^{N-1}$ sigma models, namely the form of Gauss–Weingarten equation, construction of moving frame etc. It turns out that in $\mathbb{C}P^1$ case the associated surface have constant Gauss curvature. Finally we construct explicit examples of surfaces; all but one were not constructed in this way before.

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## 2 $\mathbb{C}P^{N-1}$ sigma models and their equations of motion

The points of the complex coordinate space $\mathbb{C}^N$ will be denoted by $z = (z_1, \ldots, z_N)$ and the hermitian inner product in $\mathbb{C}^N$ by

$$\langle z, w\rangle = z^\dagger w = \sum_{j=1}^{N} \bar{z}_j w_j \,. \tag{1}$$

The complex projective space $\mathbb{C}P^{N-1}$ is defined as a set of 1–dimensional subspaces in $\mathbb{C}^N$. The manifold structure on it is defined by an open covering

$$U_k = \{[z]\ |z \in \mathbb{C}^N, z_k \neq 0\}$$

and coordinate maps

$$\varphi_k : U_k \to \mathbb{C}^{N-1}\,, \quad \varphi_k(z) = \left(\frac{z_1}{z_k}, \ldots, \frac{\hat{z}_k}{z_k}, \ldots, \frac{z_N}{z_k}\right),$$

where the symbol dash means that the element is omitted.

Let $\xi^1$, $\xi^2$ be the standard Minkowski coordinates in $\mathbb{R}^2$, with the metric

$$\mathrm{d}s^2 = (\mathrm{d}\xi^1)^2 - (\mathrm{d}\xi^2)^2\,.$$

In what follows we suppose that $\xi_L = \xi^1 + \xi^2$, $\xi_R = \xi^1 - \xi^2$ are the light–cone coordinates in $\mathbb{R}^2$, i.e.

$$\mathrm{d}s^2 = \mathrm{d}\xi_L\, \mathrm{d}\xi_R. \tag{2}$$

We shall denote by $\partial_L$ and $\partial_R$ the derivatives with respect to $\xi_L$ and $\xi_R$, respectively, i.e.

$$\partial_L = \tfrac{1}{2}\left(\partial_{\xi^1} + \partial_{\xi^2}\right)\,, \quad \partial_R = \tfrac{1}{2}\left(\partial_{\xi^1} - \partial_{\xi^2}\right)\,.$$

In the study of $\mathbb{C}P^{N-1}$ sigma models we are interested in maps

$$[z] : \Omega \to \mathbb{C}P^{N-1},$$

where $\Omega$ is an open, connected and simply connected subset in $\mathbb{R}^2$ with Minkowski metric (2), which are stationary points of the action functional (see e.g. [8])

$$\mathcal{S} = \int_\Omega \tfrac{1}{4}\,(D_\mu z)^\dagger (D^\mu z)\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,, \quad z^\dagger \cdot z = 1\,. \tag{3}$$

The covariant derivatives $D_\mu$ act on $z : \Omega \to \mathbb{C}^N$ according to the formula

$$D_\mu z = \partial_\mu z - \left(z^\dagger \cdot \partial_\mu z\right) z\,, \quad \partial_\mu \equiv \partial_{\xi^\mu}\,, \quad \mu = 1, 2 \tag{4}$$

and ensure that the action depends only on $[z] : \Omega \to \mathbb{C}P^{N-1}$ and not on the choice of a representative of the class $[z]$. Thus the map $[z]$ is determined as a solution of the Euler–Lagrange equations defined by the action (3). Writing

$$z = \frac{f}{|f|} \tag{5}$$

one may present the action functional (3) also in the form

$$\mathcal{S} = \int_\Omega \mathcal{L}\, \mathrm{d}\xi_L\, \mathrm{d}\xi_R = \int_\Omega \frac{1}{4|f|^2} \left(\partial_L f^\dagger P \partial_R f + \partial_R f^\dagger P \partial_L f\right) \mathrm{d}\xi_L\, \mathrm{d}\xi_R\,, \tag{6}$$

where the $N \times N$ matrix

$$P = 1 - \frac{1}{|f|^2}\, f \otimes f^\dagger \tag{7}$$

is an orthogonal projector on $\mathbb{C}^N$.

The equations of motion in terms of $f$ read

$$P \left\{ \partial_L \partial_R f - \frac{1}{(f^\dagger f)} \left( (f^\dagger \partial_R f) \partial_L f + (f^\dagger \partial_L f) \partial_R f \right) \right\} = 0\,, \tag{8}$$

and can be also expressed in the form of a conservation law

$$\partial_L [\partial_R P, P] + \partial_R [\partial_L P, P] = 0\,. \tag{9}$$

The real–valued currents

$$J_L = \frac{1}{f^\dagger f} \partial_L f^\dagger P \partial_L f\,, \quad J_R = \frac{1}{f^\dagger f} \partial_R f^\dagger P \partial_R f \tag{10}$$

satisfy

$$\partial_L J_R = \partial_R J_L = 0 \tag{11}$$

for any solution $f$ of the equations of motion (8).

## 3 $\mathbb{C}P^{N-1}$ sigma model and surfaces

Let us denote

$$M_L = [\partial_L P, P]\,, \quad M_R = [\partial_R P, P]\,. \tag{12}$$

Explicitly we have

$$M_D = \frac{1}{f^\dagger f} \left( P \partial_D f \otimes f^\dagger - f \otimes \partial_D f^\dagger P \right) \in su(N)\,, \quad D = L, R\,. \tag{13}$$

It follows from (13) that $\partial_B M_D \in su(N)$. From (9) we know that if $f$ is a solution of the equations of motion (8) then

$$\partial_L M_R + \partial_R M_L = 0\,. \tag{14}$$

Therefore we can make an identification

$$X_L = M_L\,, \quad X_R = -M_R \tag{15}$$

and we have a closed $su(N)$–valued 1–form

$$\mathrm{d}X = X_L\, \mathrm{d}\xi_L + X_R\, \mathrm{d}\xi_R\,. \tag{16}$$

We introduce on $su(N)$ a scalar product

$$(A, B) = -\tfrac{1}{2}\,\mathrm{tr}\,AB$$

and identify $(N^2-1)$–dimensional Euclidean space with the $su(N)$ algebra

$$\mathbb{R}^{N^2-1} \simeq su(N)\,.$$

Now we can locally associate to $\mathbb{C}P^{N-1}$ model a surface $\mathcal{F}$ by integration of the locally exact form $\mathrm{d}X$:

$$X = \int_\gamma \mathrm{d}X : \ \mathbb{R}^2 \longrightarrow \mathbb{R}^{N^2-1} \simeq su(N)\,. \tag{17}$$

The map $X$ is called the Weierstrass formula for immersion.

By computation of traces we immediately find the components of the induced metric on the surface $\mathcal{F}$

$$G = \begin{pmatrix} G_{LL}, & G_{LR} \\ G_{LR}, & G_{RR} \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned} G_{LL} &= (X_L, X_L) = \frac{1}{f^\dagger f}\,\partial_L f^\dagger P \partial_L f = J_L\,, \\ G_{LR} &= (X_L, X_R) = -\frac{1}{2f^\dagger f}\left(\partial_R f^\dagger P \partial_L f + \partial_L f^\dagger P \partial_R f\right), \\ G_{RR} &= (X_R, X_R) = \frac{1}{f^\dagger f}\,\partial_R f^\dagger P \partial_R f = J_R\,, \end{aligned}$$

i.e. the first fundamental form of the surface $\mathcal{F}$ is

$$I = J_L\,\mathrm{d}\xi_L^2 - \frac{1}{f^\dagger f}\left(\partial_R f^\dagger P \partial_L f + \partial_L f^\dagger P \partial_R f\right)\mathrm{d}\xi_L\,\mathrm{d}\xi_R + J_R\,\mathrm{d}\xi_R^2\,. \tag{19}$$

The first fundamental form $I$ defined by (19) is positive for any solution $f$ of the equations of motion (8). It is positive definite in the point $(\xi_L^0, \xi_R^0)$ either if

$$\Im\left(\partial_L f^\dagger(\xi_L^0, \xi_R^0) P \partial_R f(\xi_L^0, \xi_R^0)\right) \neq 0 \tag{20}$$

or if

$$\partial_L f(\xi_L^0, \xi_R^0)\,, \quad \partial_R f(\xi_L^0, \xi_R^0)\,, \quad f(\xi_L^0, \xi_R^0) \tag{21}$$

are linearly independent.

The conditions (20), (21) are the sufficient conditions for the existence of the surface $\mathcal{F}$ associated to the solution $f$ of the equations of motion (8) in the vicinity of the point $(\xi_L^0, \xi_R^0)$. If neither of the conditions (20), (21) is met on an image $\mathrm{Im}_X(\Theta)$ of a lower dimensional subset $\Theta \subset \mathbb{R}^2$ then the surface $\mathcal{F}$ may or may not exist depending on circumstances. If both conditions (20), (21) are violated in the whole neighborhood $\Omega \subset \mathbb{R}^2$ of the point $(\xi_L^0, \xi_R^0)$ then the surface doesn't exist in the vicinity of the point $(\xi_L^0, \xi_R^0)$.

The formula for Gaussian curvature can be written as

$$K = \frac{1}{\sqrt{G_{LL}G_{RR} - G_{LR}^2}}\,\partial_R\left(\frac{\partial_L G_{LR} - \frac{1}{2}\,G_{LR}\partial_L(\ln J_L)}{\sqrt{G_{LL}G_{RR} - G_{LR}^2}}\right). \tag{22}$$

Its explicit evaluation seems to be too complicated in general case.

In the $\mathbb{C}P^1$ case a surprising simplification happens and we have

$$K = -4\,. \tag{23}$$

Because the Gauss curvature is negative, there are no umbilical points on the surface and solutions of the equation of motion give rise to pseudospheres immersed in $su(2) \simeq \mathbb{R}^3$. Examples will be presented in Section 4.

### 3.1 Gauss−Weingarten equations

We may formally determine a moving frame on a surface in $\mathbb{R}^{N^2-1}$ and write the Gauss–Weingarten equations in the $\mathbb{C}P^{N-1}$ case.

Let $f$ be a solution of (8) such that $\det(G)$ is not zero in a neighborhood of a regular point $(\xi_L^0, \xi_R^0)$ in $\mathbb{R}^2$. Then in this neighborhood the properties of the associated surface $\mathcal{F}$ can be described by the moving frame

$$\vec{\tau} = (\partial_L X, \partial_R X, n_3, \ldots, n_{N^2-1})\,,$$

where the vectors $\partial_L X$, $\partial_R X$, $n_3$, $\ldots$, $n_{N^2-1}$ are assumed to satisfy the normalization conditions

$$\begin{gathered}(\partial_L X, \partial_L X) = G_{LL}\,, \quad (\partial_L X, \partial_R X) = G_{LR}\,, \quad (\partial_R X, \partial_R X) = G_{RR}\,,\\ (\partial_L X, n_k) = (\partial_R X, n_k) = 0\,, \quad (n_j, n_k) = \delta_{jk}\,.\end{gathered} \tag{24}$$

The moving frame satisfies the Gauss–Weingarten equations

$$\begin{aligned}\partial_L\partial_L X &= A_L^L\partial_L X + A_R^L\partial_R X + Q_j^L n_j\,,\\ \partial_L\partial_R X &= \tilde{H}_j n_j\,,\\ \partial_L n_j &= \alpha_j^L\partial_L X + \beta_j^L\partial_R X + s_{jk}^L n_k\,,\\ \partial_R\partial_L X &= \tilde{H}_j n_j\,,\\ \partial_R\partial_R X &= A_L^R\partial_L X + A_R^R\partial_R X + Q_j^R n_j\,,\\ \partial_R n_j &= \alpha_j^R\partial_L X + \beta_j^R\partial_R X + s_{jk}^R n_k\,,\end{aligned} \tag{25}$$

where $s_{jk}^L + s_{kj}^L = 0$, $s_{jk}^R + s_{kj}^R = 0$, $j, k = 3, \ldots, N^2 - 1$

$$\begin{aligned}\alpha_j^L &= \frac{\tilde{H}_j G_{LR} - Q_j^L G_{RR}}{\det G}\,, \quad & \beta_j^L &= \frac{Q_j^L G_{LR} - \tilde{H}_j G_{LL}}{\det G}\,,\\ \alpha_j^R &= \frac{Q_j^R G_{LR} - \tilde{H}_j G_{RR}}{\det G}\,, \quad & \beta_j^R &= \frac{\tilde{H}_j G_{LR} - Q_j^R G_{LL}}{\det G}\,,\end{aligned}$$

$$
\begin{aligned}
A_L^L = {} & \frac{1}{\det G}\,\Re\left\{ \frac{1}{f^\dagger f}\left(J_R\partial_L f^\dagger + G_{LR}\partial_R f^\dagger\right) P\partial_L\partial_L f \; - \right. \\
& \left. - \frac{2\partial_L f^\dagger f}{(f^\dagger f)^2}\,(\partial_R f^\dagger P \partial_L f) G_{LR} - \frac{2 f^\dagger \partial_L f}{f^\dagger f}\, J_L J_R \right\}, \\
A_R^L = {} & \frac{1}{\det G}\,\Re\left\{ -\frac{1}{f^\dagger f}\left(J_L\partial_R f^\dagger + G_{LR}\partial_L f^\dagger\right) P\partial_L\partial_L f \; + \right. \\
& \left. + \frac{2\partial_L f^\dagger f}{(f^\dagger f)^2}\,(\partial_R f^\dagger \partial_L f) J_R + \frac{2 f^\dagger \partial_L f}{f^\dagger f}\, J_L G_{LR} \right\},
\end{aligned}
\tag{26}
$$

$A_L^R$, $A_R^R$ have similar form and can be obtained by exchange $L \leftrightarrow R$. The explicit form of the coefficients $\tilde{H}_j$, $Q_j^D$ (where $D = L, R$; $j = 3, \ldots, N^2-1$) depends on the chosen orthonormal basis of the normal space $\mathrm{span}\{n_3, \ldots, n_{N^2-1}\}$ to the surface $\mathcal{F}$ at the point $X(\xi_L^0, \xi_R^0)$.

Equivalently, the Gauss–Weingarten equations can be written in the $N \times N$ matrix form

$$
\partial_L \vec{\tau} = U\vec{\tau}\,, \quad \partial_R \vec{\tau} = V\vec{\tau}\,, \tag{27}
$$

where

$$
\begin{aligned}
U = {} & \begin{pmatrix}
A_L^L & A_R^L & Q_3^L & \ldots & Q_{N^2-1}^L \\
0 & 0 & \tilde{H}_3 & \ldots & \tilde{H}_{N^2-1} \\
\alpha_3^L & \beta_3^L & s_{33}^L & \ldots & s_{3(N^2-1)}^L \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\alpha_{(N^2-1)}^L & \beta_{(N^2-1)}^L & s_{(N^2-1)3}^L & \ldots & s_{(N^2-1)(N^2-1)}^L
\end{pmatrix}, \\
V = {} & \begin{pmatrix}
0 & 0 & \tilde{H}_3 & \ldots & \tilde{H}_{N^2-1} \\
A_L^R & A_R^R & Q_3^R & \ldots & Q_{N^2-1}^R \\
\alpha_3^R & \beta_3^R & s_{33}^R & \ldots & s_{3(N^2-1)}^R \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\alpha_{(N^2-1)}^R & \beta_{(N^2-1)}^R & s_{(N^2-1)3}^R & \ldots & s_{(N^2-1)(N^2-1)}^R
\end{pmatrix}.
\end{aligned}
\tag{28}
$$

The Gauss–Codazzi–Ricci equations take the form

$$
\partial_R U - \partial_L V + [U, V] = 0 \tag{29}
$$

and are identically satisfied for any solution $f$ of (8).

The second fundamental form of the surface $\mathcal{F}$ at the regular point $p$ takes the shape of a map

$$
II(p) : \, T_p\mathcal{F} \times T_p\mathcal{F} \to N_p\mathcal{F}\,,
$$

where $T_p\mathcal{F}$, $N_p\mathcal{F}$ denotes the tangent and normal space to the surface $\mathcal{F}$ at the point $p$, respectively.

According to [9, 10], the second fundamental form can be formulated as

$$
II = \left( \frac{\partial^2 X}{\partial \xi^j \partial \xi^k} \right)^{\perp} \mathrm{d}\xi^j \, \mathrm{d}\xi^k \,,
$$

i.e.

$$II = (\partial_L\partial_L X)^\perp \mathrm{d}\xi_L\,\mathrm{d}\xi_L + (\partial_L\partial_R X)^\perp \mathrm{d}\xi_L\,\mathrm{d}\xi_R + (\partial_R\partial_R X)^\perp \mathrm{d}\xi_R\,\mathrm{d}\xi_R\,, \tag{30}$$

where $(\ )^\perp$ denotes the normal part of the vector. The mean curvature vector is

$$H = (G^{-1})^{jk}\left(\frac{\partial^2 X}{\partial\xi^j\partial\xi^k}\right)^\perp,$$

i.e.

$$H = \frac{1}{\det G}\Big(G_{RR}(\partial_L\partial_L X)^\perp - 2G_{LR}(\partial_L\partial_R X)^\perp + G_{LL}(\partial_R\partial_R X)^\perp\Big). \tag{31}$$

Unfortunately, it is clear that after explicit calculation of $(\dfrac{\partial^2 X}{\partial\xi^j\partial\xi^k})^\perp$ in the case $N > 2$ both the second fundamental form and the mean curvature vector will contain terms like $P\partial_L\partial_L f \otimes f^\dagger$ etc., since they are not in $T_p\mathcal{F}$ and they have nothing to cancel with. Therefore the resulting expressions are rather complicated and we present them only formally

$$\begin{aligned} II = {} & (\partial_L\partial_L X - A^L_L\partial_L X - A^L_R\partial_R X)\,\mathrm{d}\xi_L\,\mathrm{d}\xi_L + 2(\partial_L\partial_R X)\,\mathrm{d}\xi_L\,\mathrm{d}\xi_R + \\ & +(\partial_R\partial_R X - A^R_L\partial_L X - A^R_R\partial_R X)\,\mathrm{d}\xi_R\,\mathrm{d}\xi_R\,. \end{aligned} \tag{32}$$

where $A^L_L, \ldots, A^R_R$ are defined in (26) and $\partial_B\partial_D X$ are expressed in terms of $f$ by

$$\begin{aligned} \partial_L\partial_R X &= \partial_R\partial_L X = [\partial_L P, \partial_R P] = \\ &= \frac{1}{f^\dagger f}\left(P\partial_L f \otimes \partial_R f^\dagger P - P\partial_R f \otimes \partial_L f^\dagger P\right) + \\ &\quad + \frac{1}{(f^\dagger f)^2}\left(\partial_L f^\dagger P \partial_R f - \partial_R f^\dagger P\partial_L f\right) f \otimes f^\dagger\,, \end{aligned} \tag{33}$$

$$\begin{aligned} \partial_L\partial_L X &= \frac{1}{f^\dagger f}\left(P\partial_L\partial_L f \otimes f^\dagger - f \otimes \partial_L\partial_L f^\dagger P\right) + \\ &\quad + \frac{2}{(f^\dagger f)^2}\left((\partial_L f^\dagger f) f \otimes \partial_L f^\dagger P - (f^\dagger\partial_L f) P\partial_L f \otimes f^\dagger\right), \\ \partial_R\partial_R X &= \frac{1}{f^\dagger f}\left(f \otimes \partial_R\partial_R f^\dagger P - P\partial_R\partial_R f \otimes f^\dagger\right) + \\ &\quad + \frac{2}{(f^\dagger f)^2}\left((f^\dagger\partial_R f) P\partial_R f \otimes f^\dagger - (\partial_R f^\dagger f) f \otimes \partial_R f^\dagger P\right). \end{aligned} \tag{34}$$

The mean curvature vector is

$$\begin{aligned} H = {} & \frac{1}{\det G}\Big(G_{RR}(\partial_L\partial_L X - A^L_L\partial_L X - A^L_R\partial_R X) - 2G_{LR}(\partial_L\partial_R X) + \\ & + G_{LL}(\partial_R\partial_R X - A^R_L\partial_L X - A^R_R\partial_R X)\Big). \end{aligned} \tag{35}$$

For the $\mathbb{C}P^1$ model it is natural to identify the one–dimensional space $N_p\mathcal{F}$ with $\mathbb{R}$. Using parametrization

$$f = (1, w)\,, \tag{36}$$

we find

$$II = \frac{2\mathrm{i}}{(1+w\bar{w})^2}\left(\partial_L w \partial_R \bar{w} - \partial_R w \partial_L \bar{w}\right) \mathrm{d}\xi_L\, \mathrm{d}\xi_R\,. \tag{37}$$

### 3.2 The moving frame of a surface in the algebra $su(N)$

Let us choose an orthonormal basis in $su(N)$ in the following form

$$\begin{aligned}
(A_{jk})_{ab} &= \mathrm{i}(\delta_{ja}\delta_{kb} + \delta_{jb}\delta_{ka})\,, \quad j<k\leq N\,,\\
(B_{jk})_{ab} &= (\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka})\,, \quad j<k\leq N\,,\\
(C_p)_{ab} &= \mathrm{i}\sqrt{\frac{2}{p(p+1)}}\left(\sum_{d=1}^{p}\delta_{da}\delta_{db} - p\delta_{p+1,a}\delta_{p+1,b}\right), \quad p<N\,.
\end{aligned} \tag{38}$$

Let $f$ be a solution of the equations of motion (8) and let $(\xi_L^0, \xi_R^0)$ be a regular point in $\mathbb{R}^2$, i.e. such that $\det G\big(f(\xi_L^0,\xi_R^0)\big) \neq 0$. Let us denote $f^0 = f(\xi_L^0,\xi_R^0)$, $X^0 = X(\xi_L^0,\xi_R^0)$. Define

$$\begin{aligned}
\Phi_k^\dagger &= \begin{pmatrix}
1 & 0 & \ldots & 0 & \ldots & 0\\
0 & 1 & \ldots & 0 & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \ldots & \frac{\bar{f}_k^0}{\left(\sum_{j=k}^N f_j^0\bar{f}_j^0\right)^{1/2}} & \frac{\left(\sum_{j=k+1}^N f_j^0\bar{f}_j^0\right)^{1/2}}{\left(\sum_{j=k}^N f_j^0\bar{f}_j^0\right)^{1/2}} & 0 & \ldots\\
0 & \ldots & -\frac{\left(\sum_{j=k+1}^N f_j^0\bar{f}_j^0\right)^{1/2}}{\left(\sum_{j=k}^N f_j^0\bar{f}_j^0\right)^{1/2}} & \frac{f_k^0}{\left(\sum_{j=k}^N f_j^0\bar{f}_j^0\right)^{1/2}} & 0 & \ldots\\
0 & \ldots & 0 & 0 & 1 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \ldots & 0 & \ldots & 0 & 1
\end{pmatrix}, \quad k\leq N-2\,,\\
\Phi_{N-1}^\dagger &= \begin{pmatrix}
1 & 0 & \ldots & 0\\
0 & 1 & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots\\
0 & \ldots & \frac{\bar{f}_{N-1}^0}{\left(f_{N-1}^0\bar{f}_{N-1}^0 + f_N^0\bar{f}_N^0\right)^{1/2}} & \frac{\bar{f}_N^0}{\left(f_{N-1}^0\bar{f}_{N-1}^0 + f_N^0\bar{f}_N^0\right)^{1/2}}\\
0 & \ldots & -\frac{f_N^0}{\left(f_{N-1}^0\bar{f}_{N-1}^0 + f_N^0\bar{f}_N^0\right)^{1/2}} & \frac{f_{N-1}^0}{\left(f_{N-1}^0\bar{f}_{N-1}^0 + f_N^0\bar{f}_N^0\right)^{1/2}}
\end{pmatrix},\\
\Phi^\dagger &= \Phi_1^\dagger\Phi_2^\dagger\ldots\Phi_{N-1}^\dagger \in SU(N)\,.
\end{aligned} \tag{39}$$

If any of the denominators vanishes then the corresponding matrix $\Phi_k$ is defined to be the unit matrix.

Then

$$\begin{aligned}
\Phi^\dagger f^0 &= (\sqrt{f^{0\dagger}f^0}, 0, \ldots, 0)^T \,, \\
\partial_L^\Phi X^0 \equiv \Phi^\dagger \partial_L X(\xi_L^0, \xi_R^0)\Phi &= \frac{1}{\sqrt{f^{0\dagger}f^0}} \begin{pmatrix} 0 & -\partial_L^\Phi f^{0\dagger} \\ \partial_L^\Phi f^0 & \mathbf{0} \end{pmatrix}, \\
\partial_R^\Phi X^0 \equiv \Phi^\dagger \partial_R X(\xi_L^0, \xi_R^0)\Phi &= -\frac{1}{\sqrt{f^{0\dagger}f^0}} \begin{pmatrix} 0 & -\partial_R^\Phi f^{0\dagger} \\ \partial_R^\Phi f^0 & \mathbf{0} \end{pmatrix},
\end{aligned} \tag{40}$$

where $\mathbf{0}$ denotes null $(N-1)\times(N-1)$ matrix and the vectors $\partial_D^\Phi f^0 \in \mathbb{C}^{N-1}$ are defined by

$$(\partial_D^\Phi f^0)_{j-1} = (\Phi^\dagger \partial_D f(\xi_L^0, \xi_R^0))_j \,, \quad D = L,\, R\,, \quad j = 2, \ldots, N\,.$$

Assume that one finds orthonormal vectors (using a variant of Gramm–Schmidt orthogonalization procedure)

$$\tilde{A}_{1j}\,,\ \tilde{B}_{1j}\,, \quad j = 3, \ldots, N$$

such that

$$(\partial_K^\Phi X^0, \tilde{A}_{1j}) = 0\,, \quad (\partial_K^\Phi X^0, \tilde{B}_{1j}) = 0$$

and

$$\mathrm{span}\,(\partial_K^\Phi X^0, \tilde{A}_{1j}, \tilde{B}_{1j})_{K=L,R,\ j=3,\ldots,N} = \mathrm{span}\,(A_{1j}, B_{1j})_{j=2,\ldots,N}\,. \tag{41}$$

Define

$$\tilde{A}_{jk} = A_{jk}\,, \quad \tilde{B}_{jk} = B_{jk}\,, \quad \tilde{C}_p = C_p\,, \quad 1 < j < k \leq N\,, \quad p = 1, \ldots, N-1\,.$$

Then the moving frame of $\mathcal{F}$ at the point $X^0 = X(\xi_L^0, \xi_R^0)$ satisfying both the normalization conditions (24) and the Gauss–Weingarten equations (25) is given by

$$\begin{aligned}
\partial_L X &= \Phi \partial_L^\Phi X^0 \Phi^\dagger \,, \\
\partial_R X &= \Phi \partial_R^\Phi X^0 \Phi^\dagger \,, \\
n_{jk}^A &= \Phi \tilde{A}_{jk} \Phi^\dagger \,, \\
n_{jk}^B &= \Phi \tilde{B}_{jk} \Phi^\dagger \,, \quad 1 < j < k \leq N\,, \\
n_p^C &= \Phi \tilde{C}_p \Phi^\dagger \,, \quad p = 1, \ldots, N-1\,.
\end{aligned} \tag{42}$$

In $N = 2$ case a significant simplification occurs, namely there is only one normal vector to the surface immersed in $su(2)$

$$n = n_p^C = \Phi \tilde{C}_1 \Phi^\dagger$$

and no orthogonalization is needed.

## 4 Examples of surfaces in $su(2)$

Assuming

$$f = (1, w)\,,$$

we may write the equations of motion in the form

$$\partial_L \partial_R w - \frac{2\bar{w}}{1 + w\bar{w}} \left(\partial_L w \partial_R w\right) = 0 \tag{43}$$

and the Weierstrass coordinate representation of the surface $\mathcal{F}$ immersed in $\mathbb{R}^3$

$$\begin{aligned} X_1 &= \int_\gamma \frac{-\mathrm{i}}{2(1+w\bar{w})^2} \Big[ (\partial_L w + w^2 \partial_L \bar{w} - \partial_L \bar{w} - \bar{w}^2 \partial_L w)\, \mathrm{d}\xi_L + \\ &\quad + (\partial_R \bar{w} + \bar{w}^2 \partial_R w - \partial_R w - w^2 \partial_R \bar{w})\, \mathrm{d}\xi_R \Big], \\ X_2 &= \int_\gamma \frac{1}{2(1+w\bar{w})^2} \Big[ -(\partial_L w + w^2 \partial_L \bar{w} + \partial_L \bar{w} + \bar{w}^2 \partial_L w)\, \mathrm{d}\xi_L + \\ &\quad + (\partial_R w + w^2 \partial_R \bar{w} + \partial_R \bar{w} + \bar{w}^2 \partial_R w)\, \mathrm{d}\xi_R \Big], \\ X_3 &= \int_\gamma \frac{-\mathrm{i}}{(1+w\bar{w})^2} \Big[ (w \partial_L \bar{w} - \bar{w} \partial_L w)\, \mathrm{d}\xi_L + (\bar{w} \partial_R w - w \partial_R \bar{w})\, \mathrm{d}\xi_R \Big]. \end{aligned} \tag{44}$$

$X_i$ depend on the end point $(\xi_L, \xi_R)$ of the curve $\gamma$ in $\mathbb{R}^2$ only, its other end point $(\xi_L^0, \xi_R^0)$ is assumed to be fixed.

We remind that in the $su(2)$ case the resulting surfaces have constant negative Gaussian curvature (23)

$$K = -4\,.$$

### 4.1 Symmetry reduction

For the purpose of investigation of symmetries it appears to be useful to write the equation of motion (43) in terms of real and imaginary part. Let us denote

$$w = u + \mathrm{i}v\,.$$

Then the equation of motion (43) reads

$$\begin{aligned} \partial_L \partial_R u &= \frac{2}{1+u^2+v^2} \Big( u(\partial_L u \partial_R u - \partial_L v \partial_R v) + v(\partial_L v \partial_R u + \partial_L u \partial_R v) \Big), \\ \partial_L \partial_R v &= \frac{2}{1+u^2+v^2} \Big( u(\partial_L v \partial_R u + \partial_L u \partial_R v) - v(\partial_L u \partial_R u - \partial_L v \partial_R v) \Big). \end{aligned}$$

The algebra of symmetry generators is infinite dimensional, consisting of

$$\mathcal{G} = \mathcal{C}_{\xi_L} \oplus \mathcal{C}_{\xi_R} \oplus su(2)\,, \tag{45}$$

where $\mathcal{C}_{\xi_D}$, $D = L, R$ denote infinite dimensional algebras of conformal transformations

$$\mathcal{C}_{\xi_D} = \{ f_D(\xi_D) \partial_{\xi_D} | f_D \in \mathcal{C}^\infty(\mathbb{R}) \}$$

and $su(2)$ is generated by transformations involving only dependent coordinates

$$\begin{aligned} L_1 &= u\partial_v - v\partial_u\,, \\ L_2 &= \tfrac{1}{2}\,(1+u^2-v^2)\partial_u + uv\partial_v\,, \\ L_3 &= -uv\partial_u + \tfrac{1}{2}\,(-1+u^2-v^2)\partial_v\,. \end{aligned} \tag{46}$$

It turns out that only the vector field[1])

$$Y = L_1 + a\partial_{\xi_L} + b\partial_{\xi_R}\,, \quad a, b \in \mathbb{R} \tag{47}$$

leads to solutions allowing associated surfaces, i.e. such that (20) holds[2]). A solution invariant under (47) is of the form

$$w = R(\chi)\mathrm{e}^{\mathrm{i}(\xi_L/a - f(\chi))}\,, \quad \chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}\,, \tag{48}$$

where $R, f : \mathbb{R} \to \mathbb{R}$. Substituting this form of $w$ into the equation of motion (43) one finds two coupled ordinary differential equations of second order

$$R'' - \frac{2R}{1+R^2}R'^2 + \frac{R(1-R^2)}{1+R^2}(f' - f'^2) = 0\,, \tag{49}$$

$$f'' + \frac{1-R^2}{R(1+R^2)}(2R'f' - R') = 0\,, \tag{50}$$

$R'$, $f'$ etc. denote derivatives with respect to the symmetry variable $\chi$. The system (49),(50) has a similar form as the one obtained by the symmetry reduction of the equations of the $\mathbb{C}P^1$ sigma model in (1+2)–dimensions in [11]. Using the procedure described there we arrive at an equivalent set of ODEs

$$U'^2 = -4A^2U^4 + 4KU^3 + (8A^2 - 8K - 1)U^2 + 4KU - 4A^2\,, \tag{51}$$

$$f' = A\,\frac{(1+R^2)^2}{R^2} + \frac{1}{2}\,, \tag{52}$$

where

$$R(\chi) = \sqrt{-U(\chi)}\,, \tag{53}$$

$A$, $K$ are constants of integration.

A considerable number of solutions of (51) exists in terms of elementary functions, elliptic functions and Painlevé transcendents [11]. We now use some of them to construct several solutions of (43) and associated surfaces.

[1]) i.e. all other vector fields giving regular solutions can be transformed to (47) by symmetries
[2]) Relation (21) cannot hold if $N = 2$ since then $f$, $\partial_L f$, $\partial_R f \in \mathbb{C}^2$ are linearly dependent.

### 4.2 The tanh solution

Firstly, we select for the construction of an example of associated surface a special solution of (51)

$$U = -\tanh^2\left(\frac{\chi - c}{4a}\right).$$

Consequently, we find from (53), (52) that

$$R(\chi) = \tanh\left(\frac{\chi - c}{4a}\right), \quad f(\chi) = \frac{\chi + d}{2}, \tag{54}$$

$d \in \mathbb{R}$ being a constant of integration. Finally, substituting (54) into (48) we find the solution of the equation of motion (43)

$$w = \tanh\alpha\, \mathrm{e}^{\mathrm{i}\beta}, \tag{55}$$

where

$$\alpha = \frac{1}{4}\left(\frac{\xi_L}{a} - \frac{\xi_R}{b} - c\right), \quad \beta = \frac{1}{2}\left(\frac{\xi_L}{a} + \frac{\xi_R}{b} - d\right)$$

and $a$, $b$, $c$, $d$ are real parameters.

Using formula (14) the corresponding surface is a pseudosphere immersed in $\mathbb{R}^3$ (see Fig. 1)$^{3)}$ and can be written in a parametric form

$$\begin{aligned} X_1 &= \frac{-\cos\beta}{2\cosh 2\alpha} + \frac{1}{2\cosh 2}, \\ X_2 &= -\frac{\sin\beta}{2\cosh 2\alpha}, \\ X_3 &= \frac{\tanh 2\alpha - \tanh 2}{2} + 1 - \alpha, \end{aligned} \tag{56}$$

where we have assumed that the initial point of the curve was chosen

$$(\alpha^0, \beta^0) = (1, 0),$$

i.e.

$$(\xi_L^0, \xi_R^0) = \left(2a + \frac{d + c}{2}, -2b + \frac{d - c}{2a}\right).$$

The surface is shown in Figure (1). Its tangent and normal vectors are

$$\begin{aligned} \partial_L \vec{X} &= \left(\frac{\sin\beta + \cos\beta\tanh 2\alpha}{4a\cosh 2\alpha}, \frac{\sin\beta\tanh 2\alpha - \cos\beta}{4a\cosh 2\alpha}, -\frac{(\tanh 2\alpha)^2}{4a}\right)^T, \\ \partial_R \vec{X} &= \left(\frac{\sin\beta - \cos\beta\tanh 2\alpha}{4b\cosh 2\alpha}, \frac{\sin\beta\tanh 2\alpha + \cos\beta}{4b\cosh 2\alpha}, \frac{(\tanh 2\alpha)^2}{4b}\right)^T, \\ \vec{n} &= \left(\tanh 2\alpha\cos\beta, \tanh 2\alpha\sin\beta, \frac{1}{\cosh 2\alpha}\right)^T. \end{aligned} \tag{57}$$

$^{3)}$ All figures presented in this paper were constructed using Maple 9 computer algebra system.

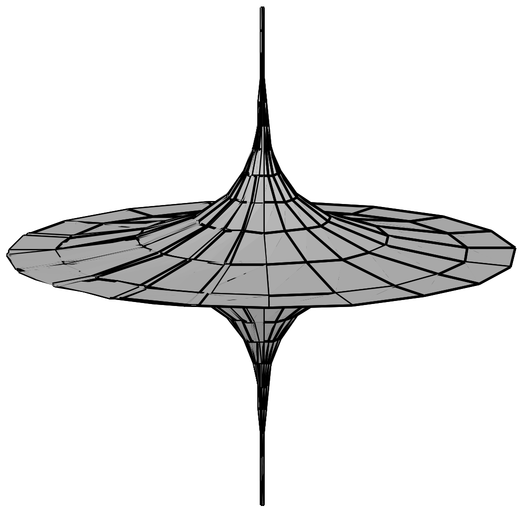

Fig. 1. Surface (56) associated to the *tanh* solution (55)

The properties of the surface are characterized by the first fundamental form

$$I = \frac{1}{16a^2}\, \mathrm{d}\xi_L\, \mathrm{d}\xi_L - \frac{1}{8ab}\left(2(\tanh 2\alpha)^2 - 1\right) \mathrm{d}\xi_L\, \mathrm{d}\xi_R + \frac{1}{16b^2}\, \mathrm{d}\xi_R\, \mathrm{d}\xi_R\,, \qquad (58)$$

the second fundamental form

$$II = \frac{1}{2ab}\, \frac{\tanh 2\alpha}{\cosh 2\alpha}\, \mathrm{d}\xi_L\, \mathrm{d}\xi_R\,, \qquad (59)$$

the principal, mean and Gaussian curvatures

$$k_1 = 2\sinh 2\alpha\,, \quad k_2 = -\frac{2}{\sinh 2\alpha}\,,$$
$$H = \sinh 2\alpha - \frac{1}{\sinh 2\alpha}\,, \quad K = -4\,. \qquad (60)$$

The values of entries in matrices $U$, $V$ in the Gauss–Weingarten equations (25) are

$$A_L^L = \frac{(\sinh 2\alpha)^2 - 1}{a \sinh 4\alpha}\,,$$
$$A_R^L = -\frac{b\cosh 2\alpha}{2a^2 \sinh 2\alpha}\,,$$

$$A_L^R = \frac{a\cosh 2\alpha}{2b^2 \sinh 2\alpha},$$
$$A_R^R = \frac{(\sinh 2\alpha)^2 - 1}{b \sinh 4\alpha}. \tag{61}$$

### 4.3 Exponential well solution

As an example of exponential solution of (49), (50) we select for the following, so-called exponential well solution

$$R(\chi) = \sqrt{\frac{(p-1)\cosh(g(\chi)) + (p+1)}{(p-1)\cosh(g(\chi)) - (p+1)}}, \tag{62}$$
$$f(\chi) = \arctan\left(\frac{p+1}{2\sqrt{-p}} \tanh g(\chi)\right) + \frac{(p + 2\sqrt{-p} - 1)\chi - 2\sqrt{-p}\chi_0}{2(p-1)} + d,$$

where

$$g(\chi) = \frac{(p+1)(\chi - \chi_0)}{2(p-1)}, \quad p < -1.$$

The solution of the equation of motion (43) is expressed using the formula (48)

$$w = R(\chi)\mathrm{e}^{\mathrm{i}(\xi_L/a - f(\chi))}, \quad \chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}.$$

The first and second fundamental forms of the associated surface can be expressed from (62) by a straightforward, if tedious calculation. Because the results appear to be quite complicated, we don't present them here. The mean curvature is

$$H = -\frac{\mathrm{e}^{4g(\chi)} - 6\mathrm{e}^{2g(\chi)} + 1}{2\,\mathrm{e}^{g(\chi)}\,(\mathrm{e}^{2g(\chi)} - 1)}. \tag{63}$$

The Weierstrass coordinate representation (44) of the associated surface in this case seems to be integrable only numerically. A picture of the surface is given in Fig. 2 for the values of parameters

$$a = 1, \quad b = 1, \quad p = -\frac{3}{2}, \quad \chi_0 = 0, \quad d = 0, \quad \xi_L, \xi_L \in (-40, \ldots, 40).$$

### 4.4 Elliptic solution

There exists also a class of solutions of (49), (50) which can be written in terms of elliptic functions. We select for the construction of a surface one of them which is written in terms of Jacobi sn function

$$R(\chi) = \sqrt{-p}\,\mathrm{sn}\left(\sqrt{Kq}\,(\chi_0 - \chi), \sqrt{\frac{p}{q}}\right), \quad f(\chi) = \frac{\chi + d}{2}, \tag{64}$$

where

$$p = \frac{1 + 8K - \sqrt{1 + 16K}}{8K}, \quad q = \frac{1 + 8K + \sqrt{1 + 16K}}{8K}$$

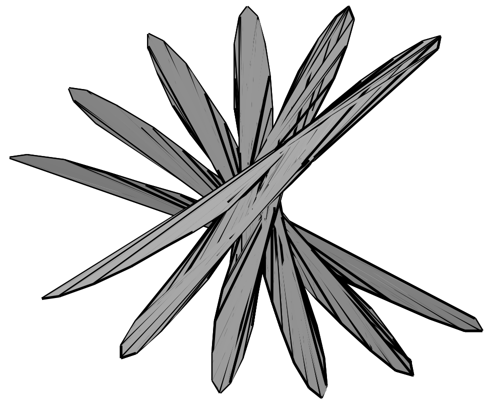

Fig. 2. The surface associated to the exponential well solution (62)

and in order for $R(\chi)$ to be real

$$K \in \left(-\frac{1}{16}, 0\right).$$

The solution of the equation of motion (43) is therefore

$$w(\xi_L, \xi_R) = \sqrt{-p}\,\mathrm{sn}\left(\sqrt{Kq}\left(\xi_0 - \frac{\xi_L}{a} + \frac{\xi_R}{b}\right), \sqrt{\frac{p}{q}}\right) \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_L/a + \xi_R/b - d)}\,. \qquad (65)$$

Since any further manipulations with the solution (65) are becoming more and more tedious and time–consuming, we resort to numerical calculation and present a picture of the associated surface in Fig. 3 for the parameters

$$a = b = 1\,, \quad \xi_0 = 0\,, \quad d = 0\,, \quad K = -\frac{1}{20}\,, \quad \xi_L, \xi_R \in (-10, \ldots, 10)\,.$$

## 5 Concluding remarks

The objective of this paper has been to present the structural equations describing two–dimensional surfaces immersed in $su(N)$ Lie algebra. We perform this analysis using the $\mathbb{C}P^{N-1}$ sigma model defined on Minkowski space. The first and second fundamental forms, the Gaussian curvature and mean curvature vector are expressed in terms of any regular solution (i.e. such that the metric (18) is non-singular) of $\mathbb{C}P^{N-1}$ sigma model. We present an implementation of this method for construction of surfaces in $su(2)$ algebra and give several examples.

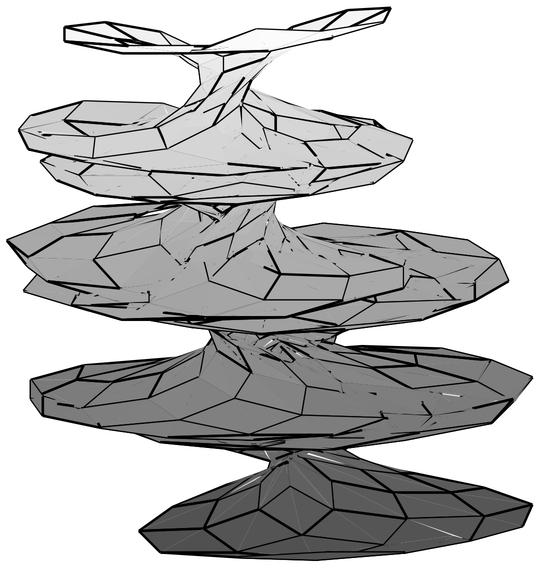

Fig. 3. The surface associated to the elliptic solution (65)

A question arises whether such approach can be extended to the complex grassmanian nonlinear sigma models in two dimensional Euclidean (Minkowski) space. Note that the grassmanian sigma models are generalization of the $\mathbb{C}P^{N-1}$ models considered here and they share various interesting properties like the existence of (anti)instantons, an infinite number of conserved quantities and complete integrability. Further, can they provide new classes of surfaces more diverse than the ones discussed in $su(2)$ case? This task shall be undertaken in our future work.

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## References


[1] C. Rogers and W.K. Schief: *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, Cambridge Univ. Press, Cambridge, 2002.

[2] A. Bobenko: *Surfaces in Terms of 2 by 2 Matrices in Harmonic Maps and Integrable Systems*, Editors A. Fordy and J. C. Wood, Vieweg, Bristol, 1990.

[3] A.S. Fokas and I.M. Gelfand: Comm. Math. Phys. **177** (1996) 203.

[4] A.S. Fokas, I.M. Gelfand, F. Finkel and Q.M. Liu: Selecta Math. New Series **6** (2000) 347.

[5] F. Helein: *Harmonic Maps, Conservation Laws and Moving Frames*, Cambridge Univ. Press, Cambridge, 2002.

[6] F. Helein: *Constant Mean Curvature Surfaces, Harmonic Maps and Integrable Systems*, Lect. Math., Birkhäuser, Boston, 2001.

[7] A.M. Grundland and L. Šnobl: preprint *math.DG/0405513*, submitted for publication.

[8] W.J. Zakrzewski: *Low Dimensional Sigma Models*, Adam Hilger, Bristol, 1989.

[9] S. Kobayashi and K. Nomizu: *Foundation of Differential Geometry*, John Wiley, New-York, 1963.

[10] T.J. Willmore: *Riemannian Geometry*, Clarendon, Oxford, 1993.

[11] A.M. Grundland, P. Winternitz and W.J. Zakrzewski: J. Math. Phys. **37** (1996) 1501.