Quantum entanglement and dynamical symmetries

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Definition of maximum entanglement in terms of a novel variational principle for quantum fluctuations and its corollaries are discussed.

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1 Introduction

It was stressed by by Schrödinger that entanglement is *"the characteristic trait of quantum mechanics"* [1]. For decades, this phenomenon was considered as a purely academic problem related to the foundation of quantum physics and touching upon the conceptual problems of reality, locality, and casuality [2, 3, 4, 5]. Recent discovery of quantum cryptography [6] and quantum teleportation [7] has led to realization that quantum entanglement is an unexpectedly efficient alternative to classical information. As a result, the quantum information science has been created as an emerging field with the potential to cause revolutionary advances in science and technology. The notion of entanglement lies at the very heart of this new science.

In spite of a great progress in investigation and implementation of entangled states, currently there is no agreement among experts on the very definition and physically motivated quantitative measure of entanglement.

The point is that the present-day conception of entanglement was formed under strong influence of information science. Consider for example the following definition, elaborated by an NSF Workshop on Quantum Information Science [8]:
"Quantum entanglement is a subtle nonlocal correlation among the parts of a quantum system that has no classical analog. Thus, entanglement is best characterized and quantified as a feature of the system that cannot be created through local operations that act on the different parts separately, or by means of classical communication among the parts".

The first key notion in this definition is the *"nonlocality"* of the systems, which indisputably is indispensable for communication and information processing. At the same time, the use of this notion leads to a loss of generality from the physical point of view. First of all, it leaves aside the single-particle entanglement that can exist at least for a single photon [9, 10, 11, 12, 13, 14, 15, 16]. Then, the requirement of nonlocality is meaningless in the case of entanglement in Bose–Einstein condensate and in ensemble of interacting fermions because of the strong overlap of wave functions of individual particles [17, 18].

Another key requirement ion the above definition is the absence of *"classical analog"* for entanglement. The fundamentally quantum nature of entanglement is usually described in terms of violation of the so-called Bell–type "classical realism"

[4, 19, 20, 21, 22, 23, 24]. Let us briefly discuss the essence of Bell's theorem.

Consider a quantum mechanical measurement of an observable X_i ($i \in I$) in a state $\psi \in \mathbf{H}_S$ of a system S. Here I is a certain set of indexes and $\mathbf{H}_S$ denotes the Hilbert space of S. According to the principles of quantum theory of measurements [25], such a measurements results in a random quantity m_i, whose probability distribution is determined by expectations of all moments $\langle\psi|(X_i)^n|\psi\rangle$ ($n = 1, 2, \cdots$). The subset of commuting observables $X_j \in \{X\}$ ($j \in J \subset I$) is specified by the joint probability distribution.

Bell's interpretation of "classical realism" consists in the assumption that all measurements, independent of whether or not they corresponds to commuting observables, have the same *hidden* joint distribution. This assumption reflects Einstein's idea of existence of *hidden variables* in quantum mechanics. Thus, violation of Bell's "classical realism" means the absence of hidden variables. In fact, such a violation signifies entanglement in the case of bipartite systems. This is caused by the simple mathematical structure of the bipartite entanglement provided by the Schmidt decomposition [26] (for modern review, see [27]). In the case of multipartite systems, Bell's conditions can be violated without manifestation of entanglement. An example of interest is provided by the so-called W state of three qubits (three spin$-1/2$ systems) [28]

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|011\rangle + |101\rangle + |110\rangle\right), \tag{1}$$

where $|\ell\ell'\ell"\rangle = |\ell\rangle \otimes |\ell'\rangle \otimes |\ell"\rangle$ ($\ell = 0, 1$). Such a state violates Bell's inequalities [29]. At the same time, it does not manifest entanglement [30].

The point is that any proper measure of entanglement should be represented by an *entanglement monotone* [31, 32]. In the case of three qubits, there is only one entanglement monotone provided by Cayley's hyperdeterminant (the *tritangle* that has been introduced in [33], is expressed in terms of Cayley's hyperdeterminant). It can be seen by direct calculation that Cayley's hyperdeterminant has zero value in the case of W–state (1). Hence this state is unentangled.

In fact, from the mathematical point of view, Bell's notion of classical realism lies within the framework of *marginal problem*, which examines conditions to have a probability density in a coordinate space with given projections onto the coordinate subspaces [34]. For more detailed discussion of connection between the marginal problem and Bell's conditions, see Ref. [35].

Thus, the above cited statement [8] cannot be considered as an adequate physical definition of entanglement.

Quite often entanglement is defined in terms of *non separability* of states in the Hilbert space of composite systems (e.g., see Ref. [36]). A counterexample of such is again provided by the unentangled W state (1), which definitely is a nonseparable one.

Probably, the most exact reflection of situation with the definition of entanglement is provided by the poetic definition by Asher Peres (for references, see [36]): *"Entanglement is a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians".*

To find the physically correct definition of entanglement, it is necessary first to separate essential from accidental. The above discussed requirements of nonlocality, violation of Bell's conditions, and nonseparability are accidental. At the same time, the requirement of "absence of classical analog" seems to be important.

By construction, the main difference between the quantum and classical levels of description of Nature consists in the choice of observables. In the former case, the observables are represented by Hermitian operators, acting in a certain Hilbert space $\mathbf{H}$. In the latter case, they are specified by c-numbers. As a result, an observable X in a state $\psi \in \mathbf{H}$ manifests *quantum fluctuations* described by the variance

$$V_X(\psi) = \langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2. \tag{2}$$

The quantum fluctuations are known to be responsible for a number of physical phenomena such as spontaneous emission, Lamb shift, and quantum jumps (see [37]).

Since existence of quantum fluctuations is the characteristic feature of quantum mechanics, it seems to be reasonable to examine the entanglement from this point of view [16, 35, 38, 39, 40, 41, 42, 43, 44, 45]. In this way, the correspondence between the level of quantum fluctuations and *maximum entanglement* has been found [16, 44]. This correspondence leads to a physically correct definition of maximum entanglement and to a number of important corollaries.

Let us stress that it is enough to define the maximum entanglement because all other entangled states can be obtained from the maximum entangled states through the use of certain local operations such as SLOCC (stochastic local operations assisted by classical communication) [28, 46, 47] and Lorentz transformations [48, 49, 50]. In other words, all entangled states of a given system belong to the same *complex orbit* [30, 35].

The investigation of maximum entanglement versus quantum fluctuations is based on an approach has been developed in Refs. [16, 35, 39, 42, 45] that can be specified as the *dynamical symmetry approach to quantum entanglement*. This approach traces back to the idea by Wigner that the general properties of a quantum mechanical system are specified by the dynamical symmetry of the corresponding Hilbert space [51, 52]. For decades, this idea has been used in quantum field theory and has demonstrated "unexpected efficiency" [52].

The aim of the present lecture is to review the main ideas of the dynamic symmetry approach to quantum entanglement and principle physical results that can be obtained within this approach. It builds upon our previous works [16, 35, 38, 39, 42, 44, 45, 53, 54, 55].

2 Essential observables

The amount of quantum fluctuations corresponding to a given state of a quantum system depends on measurements we are able to perform over the system. Among the multitude of possible observables, the set of essential or fundamental observables can be chosen in the following way.

Let S be a quantum system with the states defined in the Hilbert space $\mathbf{H}_S$ with the dynamic symmetry G. Then, the set of essential observables can be associated with the basis of the Lie algebra $\mathcal{L}$ such that $G = \exp(\mathcal{L})$.

As an example of some considerable interest, consider a system of N qubits (N "spin$-1/2$ particles"). Then, the $2N$ dimensional Hilbert space of the system

$$\mathbf{H}_{2,N} = \bigotimes_{j=1}^{N} \mathbf{H}_2 \, , \quad \dim \mathbf{H}_2 = 2 \, , \tag{3}$$

is specified by the dynamical symmetry group

$$G = \prod_{j=1}^{N} SU(2) \, . \tag{4}$$

Since observables are represented by the Hermitian operators, the *local* observables in each $\mathbf{H}_2$ should be chosen as the Pauli operators $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$, forming an infinitesimal representation of the $SL(2, \mathcal{C})$ algebra, which is known to be the *complexification* of the $SU(2)$ algebra [56]. If $|\ell\rangle$ ($\ell = 0, 1$) is the basis in $\mathbf{H}_2$, then the Pauli operators can be represented as follows

$$\begin{aligned} \sigma_x^{(j)} &= |0\rangle\langle 1| + \text{H.c.} \, , \\ \sigma_y^{(j)} &= -\mathrm{i}|0\rangle\langle 1| + \text{H.c.} \, , \\ \sigma_z^{(j)} &= |0\rangle\langle 0| - |1\rangle\langle 1| \, . \end{aligned} \tag{5}$$

In general case of *qudits* (spin$-(d-1)/2$ particles with $d \geq 2$), the essential observables are provided by the d–dimensional representation of the $SL(2, \mathcal{C})$ algebra.

It is necessary to distinguish between the qudits and d–level systems. In the latter case, the dynamical symmetry is specified by the group $SU(d)$, which leads to a higher number of essential observables. For example, a state *qutrit* ($d = 3$, spin $(d-1)/2 = 1$) is specified by the three observables represented by the three components of the spin vector, while a state of a three-level system is specified by the eight independent observables out of the nine Hermitian operators, forming representation of the $SU(3)$ algebra. In the case of Bose systems, the observables can be chosen as the generators of the Weyl–Heisenberg algebra, corresponding to the so-called quadrature operators [37]. Another possibility is connected with the representations of the simplectic subalgebras in the Weyl–Heisenberg algebra that arise, for example, in the problem of quantum description of polarization [57, 58, 59].

From the physical point of view, the selection of essential observables corresponds to the measurements we are going to perform over the system to specify its state, or, what is the same, to the Hamiltonians which are accessible for the manipulation with quantum states.

Thus, the dynamical symmetry properties of the Hilbert space of the system under consideration defines a set of corresponding essential observables.

3 Quantum fluctuations and maximum entanglement

Let $\{X\}$ be the set of essential observables of the system S defined in the Hilbert space $\mathbf{H}_S$. Then, according to Eq. (2), for each $X_i \in \{X\}$ and an arbitrary $\psi \in \mathbf{H}_S$, one can calculate the quantum fluctuation $V_i(\psi) = V_{X_i}(\psi)$. The total amount of quantum fluctuations, corresponding to the essential observables can now be chosen to specify the *remoteness* of quantum state ψ form the "classical realism", which is now understood as the measurements with classical observables (c–numbers). Thus, the remoteness of $\psi \in \mathbf{H}_S$ is

$$\mathbf{V}(\psi) = \sum_i V_i(\psi)\,. \tag{6}$$

In a sense, this is the measure of our ignorance related to the measurement of essential quantum observables in a given state.

To illustrate the meaning of this new physical quantity, consider a coherent state of a single qubit, which can be defined as follows [60]

$$|\alpha\rangle = \exp(\alpha\sigma_+ - \alpha^*\sigma_-)|1\rangle\,, \tag{7}$$

where $\sigma_+ \equiv |0\rangle\langle 1|$ and $\sigma_- \equiv |1\rangle\langle 0|$ are the generators of the $SU(2)$ algebra in two dimensions. It can be easily seen that

$$|\alpha\rangle = \mathrm{e}^{\mathrm{i}\arg\alpha}\sin|\alpha||0\rangle + \cos|\alpha||1\rangle\,. \tag{8}$$

Then, the coherence in a bipartite system can be specified by the state

$$\begin{aligned} |\alpha_1\alpha_2\rangle &= D^{(1)}(\alpha_1)D^{(2)}(\alpha_2)|11\rangle = \\ &= \mathrm{e}^{\mathrm{i}(\arg\alpha_1+\arg\alpha_2)}\sin|\alpha_1|\sin|\alpha_2||00\rangle + \mathrm{e}^{\mathrm{i}\arg\alpha_1}\sin|\alpha_1|\cos|\alpha_2||01\rangle + \\ &\quad +\mathrm{e}^{\mathrm{i}\arg\alpha_2}\sin|\alpha_2\cos|\alpha_1||10\rangle + \cos|\alpha_1|\cos|\alpha_2||11\rangle\,, \end{aligned} \tag{9}$$

where $D^{(j)}(\alpha_j) \equiv \exp\left(\alpha_j\sigma_+^{(j)} - \alpha_j^*\sigma_-^{(j)}\right)$ is the displacement operator in the two–dimensional Hilbert space of a qubit. It is now a straightforward matter to show that for all complex α_1 and α_2 the total fluctuation (remoteness) (6) in the coherent state (9) has the form

$$\mathbf{V}(\alpha_1, \alpha_2) = 4\,.$$

As all one can expect, this is the minimal value of remoteness for quantum states of two qubits. As a matter of fact, a general coherent state always has minimal amount of quantum fluctuations and therefore is considered as next to the classical state [61, 62].

In turn, in the case of maximum entangled Bell state of two qubits

$$|\psi_{\mathrm{Bell}}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \tag{10}$$

the remoteness achieves the maximum value $\mathbf{V}(\mathrm{Bell}) = 6$. This fact agrees with the statement that entanglement is a fundamental quantum property without classical analog. It is possible to check that many other maximum entangled states obey the same condition of maximum remoteness [39].

Therefore, we choose to define the maximum entangled state $\psi_{\mathrm{ME}} \in \mathbf{H}_S$ of an arbitrary quantum system S by the condition [16, 44, 45]

$$\mathbf{V}\left(\psi_{\mathrm{ME}}\right) = \max_{\psi \in \mathbf{H}_S} \mathbf{V}(\psi) \,. \tag{11}$$

It is seen that in fact Eq. (11) represents a *novel variational principle in quantum mechanics*. In a sense, (11) is similar to the principle of maximum entropy in quantum statistical mechanics, specifying the equilibrium states of quantum systems [63]. It is appropriate mention here that the notion of *skew information* has been introduced by Wigner and Yanase [64] is also based on the amount of quantum fluctuations peculiar to a quantum state. Similar quantity is also used in the estimation of mean error in the standard process of quantum state reconstruction [65].

Let us stress that the amount of quantum fluctuations (remoteness) cannot be used as the measure of entanglement as well as the entropy cannot be used as a measure of deviation from the equilibrium state in statistical mechanics. For example, the unentangled W state (1) has quite high remoteness $\mathbf{V}(W) = 8 + 2/3$ ($\mathbf{V}_{\max} = 9$ for a tree–qubit system), while certain entangled states of three qubits have less amount of remoteness [44].

Usually, expression of physical properties in the succinct and elegant form of variational principles has many advantages. Below we consider some corollaries of the variational principle (11).

4 Corollaries of the variational principle (11)

4.1 Corollary 1: Physical meaning of Maximum Entanglement

From the physical point of view, the maximum entangled state defined by Eq. (11) represents *the manifestation of quantum fluctuations at their extreme*. This definition aligns the maximum entanglement with the known physical phenomena like coherence and squeezing, whose behavior is also specified by the amount of quantum fluctuations [37]. In particular, the maximum entanglement is, by definition, an exact antithesis with respect to the notion of coherence.

Let us stress that the single–mode squeezed state

$$|\xi\rangle = \mathrm{e}^{(\xi^* a^2 - \xi a^{+2})/2} |\mathrm{vac}\rangle \tag{12}$$

can be considered as a kind of parametric maximum entangled state. In this case, the observables are provided by the quadrature operators

$$q = \frac{1}{2}\left(a + a^+\right), \quad p = \frac{-\mathrm{i}}{2}\left(a - a^+\right), \quad \left[a, a^+\right] = 1\,,$$

and the remoteness has the form

$$\mathbf{V}(\xi) = \frac{1}{2}\left(2\cosh r - 1\right), \tag{13}$$

where $\xi = r\exp(\mathrm{i}\theta)$. At the same time,

$$\bar{n} = \langle\xi|a^+a|\xi\rangle = \sinh^2 r\,,$$

so that

$$\mathbf{V}(\xi) = \bar{n} + 1/2\,. \tag{14}$$

At the same time, in the case of coherent state $\mathbf{V}(\text{coherent}) = 1/2$, which corresponds to the minimal remoteness.

4.2 Corollary 2: Expectation values of observables

Assume now that the essential observables $\{X\}$ form a representation of a compact Lie algebra. In this case, there is the uniquely defined Casimir operator of the form

$$\hat{C} = \sum_i X_i^2 = C \otimes \mathbf{1}\,, \tag{15}$$

where $\mathbf{1}$ denotes the unit operator. Then it follows from the definition of quantum fluctuation (2) and remoteness (6) that the variational principle (11) leads to the equivalent condition

$$\forall X_i \in \{X\} \quad \langle\psi_{\mathrm{ME}}|X_i|\psi_{\mathrm{ME}}\rangle = 0\,. \tag{16}$$

Under this condition

$$\mathbf{V}(\psi_{ME}) = C\,.$$

This condition has been proposed in [39] as an *operational* definition of maximum entanglement (definition in terms of what can be measured) valid for the systems with observables represented by generators of the compact Lie algebras.

It can be easily seen that condition (16) shows the existence of infinitely many maximum entangled states in the system of qudits at $d \geq 2$. In fact, in the case of N qudits, the number of equations (16) is $3N$. One more equation comes from the normalization condition $\langle\psi_{\mathrm{ME}}|\psi_{\mathrm{ME}}\rangle = 1$, so that we have $(3N+1)$ equations altogether to determine the parameters of the wave function ψ_{ME}. The latter is specified by the $2 \times d^N$ real coefficients. It is seen that for all $d \geq 2$, the number of coefficients exceeds the number of equations, so that we get infinitely many solutions corresponding to the maximum entangled state. An exception is provided by the case of $d = 2$ and $N = 1$, when the number of equations $3N + 1 = 4$ coincides with the number of real coefficients, specifying the wave function. It can be easily checked that Eqs. (16) have only trivial solutions in this case, so that the maximum entangled state of a single qubit does not exist.

It should be emphasized that among the possible maximum entangled states of a given system only states, forming a basis of $\mathbf{H}_S$ are important. Just these states are used in the teleportation, for example. The procedure, how to construct the basis of maximum entangled states for an arbitrary qudit system has been described in [44].

4.3 Corollary 3: Single−particle entanglement

It is clear that the variational principle (11) defines maximum entanglement irrespective of physical realization of the system S. In other words, this variational principle strongly extends the range of application of entanglement has been restricted so far by the framework of quantum information science.

As an illustrative example of some considerable interest, let us examine the maximum entangled states of a single qutrit. If $|\ell\rangle$ ($\ell = 0, 1, 2$) denotes the base vectors of the three–dimensional Hilbert space $\mathbf{H}_3$, then an arbitrary pure state of qutrit can be written as follows

$$|\psi\rangle = \sum_{\ell=0}^{2} \psi_\ell |\ell\rangle \,, \quad \sum_{\ell=0}^{2} |\psi_\ell|^2 = 1 \,. \tag{17}$$

Taking into account that observables in this case have the form

$$\begin{aligned} S_x &= \frac{1}{\sqrt{2}} \big(|0\rangle\langle 1| + |1\rangle\langle 2| + \mathrm{H.c.} \big) \,, \\ S_y &= \frac{-\mathrm{i}}{\sqrt{2}} \big(|0\rangle\langle 1| + |1\rangle\langle 2| - \mathrm{H.c.} \big) \,, \\ S_z &= |0\rangle\langle 0| - |2\rangle\langle 2| \,, \end{aligned}$$

The condition (16) then gives the following equations

$$\begin{aligned} \mathrm{Re}\big(\psi_0\psi_1^*\big) + \mathrm{Re}\big(\psi_1\psi_2^*\big) &= 0 \,, \\ \mathrm{Im}\big(\psi_0\psi_1^*\big) + \mathrm{Re}\big(\psi_1\psi_2^*\big) &= 0 \,, \\ |\psi_0|^2 - |\psi_2|^2 &= 0 \,. \end{aligned} \tag{18}$$

It is now straightforward to see that Eqs. (18) together with the normalization condition in (17) have infinitely many nontrivial solutions, specifying maximum entangled states of a single qutrit. In particular, the states

$$|\psi_1\rangle = |1\rangle \,, \quad |\psi_\pm\rangle = \frac{1}{\sqrt{2}} \big(|0\rangle \pm |2\rangle \big) \tag{19}$$

form the basis in the Hilbert space of qutrit. If qutrit is associated with either spin or angular momentum $j = 1$, then the states (19) can be interpreted in terms of a given projection m as follows:

$$|\psi_1\rangle = |j=1, m=0\rangle \,, \quad |\psi_\pm\rangle = \frac{1}{\sqrt{2}} \big(|j=1, m=1\rangle \pm |j=1, m=-1\rangle \big) \,. \tag{20}$$

Thus, the state $|j = 1, m = 0\rangle$ is the maximum entangled state.

A simple example of a qutrit particle is provided by a dipole photon, having angular momentum $j = 1$. In this case, the observables should be rewritten in terms of the spin subalgebra in the Weyl-Heisenberg algebra of dipole photons a_m as follows [57, 66]

$$\begin{aligned} J_x &= \left[a_0^+(a_+ + a_-) + \text{H.c.}\right]/\sqrt{2}\,, \\ J_y &= \left[\mathrm{i}a^+(a_+ - a_-) + \text{H.c.}\right]/\sqrt{2}\,, \\ J_z &= a_+^+a_+ - a_-^+a_-\,. \end{aligned}$$

It can be easily seen now that the remoteness of the single–photon state $|1_{(j=1,m=0)}\rangle$, corresponding to the maximum entangled state $|\psi_1\rangle$ in (19), has the maximum value: $\mathbf{V}\big(|1_{(j=1,m=0)}\rangle\big) = 2$. The two other states with given projection of the total angular momentum have lesser amount of quantum fluctuations: $\mathbf{V}\big(|1_{(j=1,m=\pm)}\rangle\big) = 1$. Since the angular momentum of photons can be measured [67], the different amount of quantum fluctuations of different states can be observed experimentally.

The fact that the angular momentum of a photon consists of the spin (polarization) and orbital (azimuthal phase) parts, makes it possible to consider it as a local system of two qubits. One qubit is provided by the two helicities, while another qubit corresponds to the orbital angular momentum that can be measured as well. It should be stressed that the electric dipole photon always has only two different values of the orbital angular momentum and that its vector potential (wave function) corresponds to an entangled state [68].

Similar two–qubit structure takes place in the case of particles of quite different physical nature, namely in the case of π–mesons. According the the modern theory [69], these particles consist of the “up” and “down” quarks. Since each quark can be observed in two states, it can be considered as a qubit. The $\pi^{\pm}$–mesons represent the coherent states of quarks

$$\pi^+ = u\bar{d}\,, \quad \pi^- = \bar{u}d\,.$$

In contrast, π^0–meson is prepared in the maximum entangled state of two qubits

$$\pi^0 = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}\,.$$

Since the maximum entangled state corresponds to the maximal level of quantum fluctuations, π^0–meson should be less stable that $\pi^{\pm}$–mesons. This conclusion agrees with the fact that the ratio of lifetimes $\tau_0/\tau_{\pm} \sim 10^{-9}$.

There is also a strong similarity between the triplet of π–mesons and superfluid ^{3}He. In particular, the so-called A–phase with projection of angular momentum $m = 0$ [70] is similar to the state of π^0–meson and hence is the maximum entangled state as well.

4.4 Corollary 4: Stabilization of entanglement

Different applications in quantum information processing and quantum computing require not an arbitrary entanglement but a *robust* one. This assumes quite high amount of entanglement and long enough life time. The variational principle (11) clarifies how to prepare robust entanglement. Namely, as the first step, the state of the system S with maximum amount of quantum fluctuations should be prepared. Then, the energy of the system should be decreased up to a (local) minimum under the condition of conservation of the level of quantum fluctuations. Thus, the stabilization of entanglement requires a certain minimax procedure.

Physically, this procedure can be realized through the use of interaction between the system and specially selected dissipative environment. Some examples were considered in Refs. [53, 54, 55].

5 Conclusion

Thus, we have shown that the maximum entanglement can be defined in terms of the variational principle (11) as the manifestation of quantum fluctuations at their extreme. All other entangled states are equivalent to the maximum entangled state to within a certain local transformations. The new definition of entanglement, based on the variational principle (11) is free of accidental assumptions like nonlocality, nonseparability, and violation of Bell's conditions.

This variational principle (11) treats entanglement as a physical phenomenon and hence strongly extends the range of application of the notion of entanglement. In particular, it makes it possible to consider the single–particle entanglement with respect to intrinsic degrees of freedom. This, in turn, can shed a light on the problem of stability of elementary particles.

The variational principle (11) opens the physical way of stabilization of entanglement for further applications.

Our consideration so far has been connected with the pure quantum states. It can be easily generalized on the case of mixed states because the latter can be treated as pure states of a certain "doublet", consisting of the system S and its "mirror image" [71].

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