Non-trivial extension of the Poincaré algebra for antisymmetric gauge fields

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We investigate a non-trivial extension of the D-dimensional Poincaré algebra. Matrix representations are obtained. The bosonic multiplets contain antisymmetric tensor fields. It turns out that this symmetry acts in a natural geometric way on these p–forms. Some field theoretical aspects of this symmetry are studied and invariant Lagrangians are explicitly given.

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1 Introduction

The usual electromagnetic gauge field admits a generalization in terms of antisymmetric pth order tensors or p–forms. The revival of interest for the p–forms is mainly due to there appearance in supergravity or string theory. Furthermore, the pth antisymmetric gauge fields naturally couple to $(p-1)$–dimensional extended objects. However, the p–forms are known to be relatively rigid in the sense that there is a few number of consistent interactions for them [1]. Despite these restrictions p–forms may present interesting symmetry properties, as it is the case, for instance, with the duality transformations between types IIA and IIB string theories [2]. In this paper we investigate a new possible symmetry among p–forms. This symmetry turns out to be a non-trivial extension of the Poincaré algebra in D–dimensional space–time, an extension different from supersymmetry.

By definition this symmetry is not in contradiction with the no-go theorems [3] because its underlying algebraic structure is neither a Lie algebra nor a Lie superalgebra, but an $F-$Lie algebra. The F–Lie algebras were introduced in [4, 5]. They admit a $\mathbb{Z}_F$–gradation, the zero–graded part being a Lie algebra and the non-zero graded parts being appropriate representations of the zero–graded part. An F–fold symmetric product (playing the role of the anticommutator in the case $F=2$) expresses the zero graded part in terms of the non-zero graded part. In

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particular when the zero–graded part coincides with the Poincaré algebra, one has additional generators, denoted V, such that $V^F \sim P$, with P the space–time translation generators. Then, it has been realized that, within the framework of F–Lie algebras, different extensions of the Poincaré algebra can be constructed.
Firstly, one considers the new generators V belonging to an infinite dimensional representation of the Poincaré algebra (a Verma module) [4]. Such extensions in $(1+2)$–dimensional space–time have been considered, with V belonging to the spin $1/F$–representation of the $(1+2)D$ Poincaré group. It has been shown that this symmetry acts on relativistic anyons [6] and can be seen as a direct generalisation of supersymmetry, named fractional supersymmetry [7].
Secondly, finite dimensional F–Lie algebras have been obtained by an inductive theorem, starting from any Lie (super)algebras [5]. Inönü–Wigner contractions of certain of them allow other types of extensions of the Poincaré algebra.

Following this second line, in a previous paper [8] a field theoretical realization of the simplest extension (with $F = 3$) has been realized in four space–time dimensions. This new extension of the Poincaré algebra was named cubic supersymmetry or 3SUSY algebra. Since this extension is based on some new generators in the vector representation of the Poincaré algebra (see bellow), its representations contain states of a definite statistics, *i.e.* the multiplets are purely bosonic or purely fermionic. Thus, the name cubic supersymmetry may be misleading. In order to avoid confusion, from now on, this symmetry will be called cubic symmetry.

In this paper, we will extend to an arbitrary number of space–time dimensions some of the results obtained in [8]. Studying the representation theory of this algebra, one obtains, among other possibilities, antisymmetric tensor multiplets containing p–forms. We thus have a symmetry that acts naturally on these p–forms.

In the next section, we recall briefly the underlying algebra and its matrix representation in D–space–time dimensions. Then, we explicitly consider the antisymmetric tensor multiplets and their transformation laws in $D = 1 + (4n - 1)$. These transformations acting on p–forms have an interesting geometrical interpretation in terms of the inner and exterior product with a one–form, the parameter of the transformation. In section 3, invariant Lagrangians are obtained. We observe that cubic symmetry invariance requires gauge fixing terms adapted for p–forms thus generalizing the usual gauge fixing term of electrodynamics. In the last section some conclusions are given.

2 Non-trivial extension of the Poincaré algebra

In this section we will extend to an arbitrary number of space–time dimensions the cubic symmetry algebra and its matrix representation. Antisymmetric tensor multiplets and their transformations laws will then be obtained.

2.1 The algebra and matrix representations

The cubic symmetry algebra is constructed from the Poincaré generators (L_{MN} and P_M, $M,N=0,\ldots,D-1$) together with some additional generators V_M in the vector representation of the Poincaré algebra,

$$\begin{aligned}
&[L_{MN},L_{PQ}]=\eta_{NP}L_{PM}-\eta_{MP}L_{PN}+\eta_{NP}L_{MQ}-\eta_{MP}L_{NQ}\,,\\
&[L_{MN},P_P]=\eta_{NP}P_M-\eta_{MP}P_N\,,\\
&[L_{MN},V_P]=\eta_{NP}V_M-\eta_{MP}V_N\,,\quad [P_M,V_N]=0\,,\\
&\{V_M,V_N,V_R\}=\eta_{MN}P_R+\eta_{MR}P_N+\eta_{RN}P_M\,,
\end{aligned}\tag{1}$$

where $\{V_M,V_N,V_P\}=V_MV_NV_R+V_MV_RV_N+V_NV_MV_R+V_NV_RV_M+V_RV_MV_N+V_RV_NV_M$ stands for the symmetric product of order 3 and $\eta_{MN}=\mathrm{diag}\,(1,-1,\ldots,-1)$ is the D–dimensional Minkowski metric. A matrix representation of our algebra is given by

$$V_M=\begin{pmatrix}0&\Lambda^{1/3}\Gamma_M&0\\0&0&\Lambda^{1/3}\Gamma_M\\\Lambda^{-2/3}P_M&0&0\end{pmatrix}\tag{2}$$

with Γ_M the D–dimensional Γ–matrices ($\{\Gamma_M,\Gamma_N\}=2\eta_{MN}$), $P_M=\partial_M$[1]) and Λ a parameter with mass dimension. When D is an even number ($D=2k$) the Γ–matrices can be written as $\Gamma_M=\begin{pmatrix}0&\Sigma_M\\\tilde\Sigma_M&0\end{pmatrix}$, with $\Sigma_0=\tilde\Sigma_0=1$, $\tilde\Sigma_I=-\Sigma_I$, $I=1,\ldots,D-1$ and Σ_I the generators of the Clifford algebra $SO(2k-1)$, $\Sigma_I\Sigma_J+\Sigma_J\Sigma_I=2\delta_{IJ}$. Thus, the representation (2) is reducible leading to two inequivalent representations:

$$\begin{aligned}
V_{+M}&=\begin{pmatrix}0&\Lambda^{1/3}\Sigma_M&0\\0&0&\Lambda^{1/3}\tilde\Sigma_M\\\Lambda^{-2/3}P_M&0&0\end{pmatrix},\\
V_{-M}&=\begin{pmatrix}0&\Lambda^{1/3}\tilde\Sigma_M&0\\0&0&\Lambda^{1/3}\Sigma_M\\\Lambda^{-2/3}P_M&0&0\end{pmatrix}.
\end{aligned}\tag{3}$$

For simplicity, we set from now on $\Lambda=1$ (in appropriate units). It has been noticed before that particles in an irreducible representation of (1) are degenerate in mass [8], because P^2 is a Casimir operator.

For further use we introduce the antisymmetric set of Γ matrices

$$\Gamma^{(\ell)}:\Gamma_{M_1\cdots M_\ell}=\frac{1}{\ell!}\sum_{\sigma\in S_\ell}\Gamma_{M_{\sigma(1)}}\cdots\Gamma_{M_{\sigma(\ell)}}\tag{4}$$

(with S_ℓ the group of permutations with ℓ elements) which for even D gives a further simplification

$$\Gamma_{M_1\cdots M_{2\ell}}=\begin{pmatrix}\Sigma_{M_1}\tilde\Sigma_{M_2}\cdots\Sigma_{M_{2\ell-1}}\tilde\Sigma_{M_{2\ell}}+\ \text{perm}&0\\0&\tilde\Sigma_{M_1}\Sigma_{M_2}\cdots\tilde\Sigma_{M_{2\ell-1}}\Sigma_{M_{2\ell}}+\ \text{perm}\end{pmatrix}$$

[1]) It should be noted that, in this paper, we take the structure constants to be real — see Eq.(1)— and consequently there is no i factor in P_M.

$$
\begin{aligned}
&= \begin{pmatrix} \Sigma_{M_1\cdots M_{2\ell}} & 0 \\ 0 & \tilde{\Sigma}_{M_1\cdots M_{2\ell}} \end{pmatrix}, \\
\Gamma_{M_1\cdots M_{2\ell+1}} &= \begin{pmatrix} 0 & \Sigma_{M_1}\tilde{\Sigma}_{M_2}\cdots\tilde{\Sigma}_{M_{2\ell}}\Sigma_{M_{2\ell+1}} + \text{ perm} \\ \tilde{\Sigma}_{M_1}\Sigma_{M_2}\cdots\Sigma_{M_{2\ell}}\tilde{\Sigma}_{M_{2\ell+1}} + \text{ perm} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Sigma_{M_1\cdots M_{2\ell+1}} \\ \tilde{\Sigma}_{M_1\cdots M_{2\ell+1}} & 0 \end{pmatrix},
\end{aligned} \tag{5}
$$

where the definitions of the Σ, $\tilde{\Sigma}$ matrices can be deduced from the equalities above. For instance, $\Sigma_{M_1\ldots M_{2\ell}} = \sum\limits_{\sigma\in S_\ell} \Big(\prod\limits_{i=1}^{\ell} \Sigma_{M_{\sigma(2i-1)}}\tilde{\Sigma}_{M_{\sigma(2i)}}\Big)$ and similarly for the other matrices.

2.2 Antisymmetric tensor multiplets

To build a representation of the algebra (1) using the matrix representations (2) and (3) we have also to specify the representation of the vacuum. If the vacuum is in the trivial representation of the Lorentz algebra, the multiplet consists of three spinors $\mathbf{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$ transforming as $\delta_\varepsilon \mathbf{\Psi} = \varepsilon^M V_M \mathbf{\Psi}$. When the space–time dimension is even, this representation is reducible and consist of three spinors of definite chirality. We have two possibilities corresponding to the two choices for the matrices $V_\pm$ (see (3)) $\mathbf{\Psi}_+ = \begin{pmatrix} \Psi_{1+} \\ \bar{\Psi}_{2-} \\ \Psi_{3+} \end{pmatrix}$ or $\mathbf{\Psi}_- = \begin{pmatrix} \bar{\Psi}_{1-} \\ \Psi_{2+} \\ \bar{\Psi}_{3-} \end{pmatrix}$. They transform as $\delta_\varepsilon \mathbf{\Psi}_\pm = \varepsilon^M V_{\pm M} \mathbf{\Psi}_\pm$, with ε a *commuting Lorentz vector* that we take real (we stress that ε is not an anticommuting spinor unlike in supersymmetry), Ψ_+ a left–handed spinor and $\bar{\Psi}_-$ a right–handed spinor.

From now on we consider the case of $D = 1 + (4n-1)$ space–time dimensions, the general case will be studied elsewhere. We concentrate on the case where the vacua are in the spinor representations of the Lorentz algebra. We take two copies $\mathbf{\Psi}_\pm, \mathbf{\Lambda}_\pm$ transforming with $V_\pm$, and two copies of the vacuum in the spinor representation $\Omega_\pm$, $\omega_\pm$. From the decomposition of the product of spinors on the set of p–forms

$$
\begin{aligned}
\mathcal{S}_+ \otimes \mathcal{S}_+ &= [0] \oplus [2] \oplus \cdots [2n]_+ \,, \\
\mathcal{S}_- \otimes \mathcal{S}_- &= [0] \oplus [2] \oplus \cdots [2n]_- \,, \\
\mathcal{S}_+ \otimes \mathcal{S}_- &= [1] \oplus [3] \oplus \cdots [2n-1]
\end{aligned} \tag{6}
$$

with $\mathcal{S}_\pm$ a left/right handed spinor and $[p]$ representing a p–form and $[2n]_\pm$ an (anti–)self–dual $2n$–form, one gets the four multiplets

$$
\begin{aligned}
\Xi_{++} &= \mathbf{\Psi}_+ \otimes \Omega_+ = \begin{pmatrix} \Xi_{1++} \\ \bar{\Xi}_{2-+} \\ \Xi_{3++} \end{pmatrix} = \begin{pmatrix} A_{[0]} \oplus A_{[2]} \oplus \cdots \oplus A_{[2n]_+} \\ \tilde{A}_{[1]} \oplus \tilde{A}_{[3]} \oplus \cdots \oplus \tilde{A}_{[2n-1]} \\ \tilde{\tilde{A}}_{[0]} \oplus \tilde{\tilde{A}}_{[2]} \oplus \cdots \oplus \tilde{\tilde{A}}_{[2n]_+} \end{pmatrix}, \\
\Xi_{--} &= \mathbf{\Psi}_- \otimes \Omega_- = \begin{pmatrix} \bar{\Xi}_{1--} \\ \Xi_{2+-} \\ \bar{\Xi}_{3--} \end{pmatrix} = \begin{pmatrix} A'_{[0]} \oplus A'_{[2]} \oplus \cdots \oplus A'_{[2n]_-} \\ \tilde{A}'_{[1]} \oplus \tilde{A}'_{[3]} \oplus \cdots \oplus \tilde{A}'_{[2n-1]} \\ \tilde{\tilde{A}}'_{[0]} \oplus \tilde{\tilde{A}}'_{[2]} \oplus \cdots \oplus \tilde{\tilde{A}}'_{[2n]_-} \end{pmatrix}, \\
\Xi_{-+} &= \mathbf{\Lambda}_- \otimes \omega_+ = \begin{pmatrix} \xi_{1-+} \\ \bar{\xi}_{2++} \\ \xi_{3-+} \end{pmatrix} = \begin{pmatrix} A_{[1]} \oplus A_{[3]} \oplus \cdots \oplus A_{[2n-1]} \\ \tilde{A}_{[0]} \oplus \tilde{A}_{[2]} \oplus \cdots , \tilde{A}_{[2n]_+} \\ \tilde{\tilde{A}}_{[1]} \oplus \tilde{\tilde{A}}_{[3]} \oplus \cdots \oplus \tilde{\tilde{A}}_{[2n-1]} \end{pmatrix}, \\
\Xi_{+-} &= \mathbf{\Lambda}_+ \otimes \omega_- = \begin{pmatrix} \bar{\xi}_{1+-} \\ \xi_{2--} \\ \bar{\xi}_{3+-} \end{pmatrix} = \begin{pmatrix} A'_{[1]} \oplus A'_{[3]} \oplus \cdots \oplus A'_{[2n-1]} \\ \tilde{A}'_{[0]} \oplus \tilde{A}'_{[2]} \oplus \cdots \oplus \tilde{A}'_{[2n]_-} \\ \tilde{\tilde{A}}'_{[1]} \oplus \tilde{\tilde{A}}'_{[3]} \oplus \cdots \oplus \tilde{\tilde{A}}'_{[2n-1]} \end{pmatrix}.
\end{aligned}
\tag{7}
$$

Thus in each Ξ multiplets we have various set of p–forms, with $0 \leq p \leq 2n$. Due to the property of (anti–)self–duality of $2n$–forms in $1+(4n-1)$–dimensions, ${}^\star A_{[2n]_+} = \mathrm{i} A_{[2n]_+}$, ${}^\star A'_{[2n]_-} = -\mathrm{i} A'_{[2n]_-}$, *etc* (with ${}^\star A_{[2n]_+}$ the Hodge dual of $A_{[2n]_+}$), the $2n$–forms are complex representations of $SO(1, D-1)$ and consequently also the other p–forms (see Eq.[14] below). The multiplets in (7) can be taken complex conjugate of each other ($\Xi^\star_{++} = \Xi_{--}$ and $\Xi^\star_{+-} = \Xi_{-+}$), that is

$$
\begin{aligned}
&A^\star_{[2p]} = A'_{[2p]}\,, \qquad A^\star_{[2n]_+} = A'_{[2n]_-}\,, \\
&\tilde{\tilde{A}}^\star_{[2p]} = \tilde{\tilde{A}}'_{[2p]}\,, \qquad \tilde{\tilde{A}}^\star_{[2n]_+} = \tilde{\tilde{A}}'_{[2n]_-}\,, \\
&\tilde{A}^\star_{[2p+1]} = A'_{[2p+1]}
\end{aligned}
\tag{8}
$$

(the complex conjugate $A^\star$ of A should not to be confused with its dual ${}^\star A$) and similarly for the fields coming from Ξ_{-+} and Ξ_{+-}.

The underlying algebra (1) and its representations (2) and (3) have a $\mathbb{Z}_3$–graded structure. Hence, one can assume that the fields with no tilde symbol are in the (-1)–graded sector, the fields with one tilde are in the 0–graded sector and the fields with two tilde symbols are in the 1–graded sector. For example, for the multiplet Ξ_{++}, the fields $A_{[0]}, \ldots, A_{[2n]_+}$ are in the (-1)–graded sector, the fields $\tilde{A}_{[1]}, \ldots, \tilde{A}_{[2n-1]}$ are in the 0–graded sector and the fields $\tilde{\tilde{A}}_{[0]}, \ldots, \tilde{\tilde{A}}_{[2n]_+}$ are in the 1–graded sector.

Now, in order to obtain the transformation laws of the p–forms, we give first the general relations which allow to project out any p–form $\omega_{[p]}$ from the product of two spinors, denoted $\Psi_\pm \otimes \Psi_\pm$. One has

$$\begin{aligned}
\Psi_{++} &= \Psi_+ \otimes \Psi'^t_+ \mathcal{C}_+ = \sum_{p=0}^{n-1} \frac{1}{(2p)!} \omega_{[2p]} \Sigma^{(2p)} + \frac{1}{2}\frac{1}{(2n)!} \omega_{[2n]_+} \Sigma^{(2n)}, , \\
\Psi_{--} &= \Psi_- \otimes \Psi'^t_- \mathcal{C}_- = \sum_{p=0}^{n-1} \frac{1}{(2p)!} {\omega'}_{[2p]} \tilde{\Sigma}^{(2p)} + \frac{1}{2}\frac{1}{(2n)!} {\omega'}_{[2n]_-} \tilde{\Sigma}^{(2n)} , \\
\Psi_{+-} &= \Psi_+ \otimes \Psi'^t_- \mathcal{C}_- = \sum_{p=0}^{n-1} \frac{1}{(2p+1)!} \omega_{[2p+1]} \Sigma^{(2p+1)} , \\
\Psi_{-+} &= \Psi_- \otimes \Psi'^t_+ \mathcal{C}_+ = \sum_{p=0}^{n-1} \frac{1}{(2p+1)!} {\omega'}_{[2p+1]} \tilde{\Sigma}^{(2p+1)} .
\end{aligned} \tag{9}$$

Since the Σ matrices (5) act on spinors, they have the first spinor index up and the second spinor index down. This means that in the equations above the index of the second spinor has to be raised by means of the charge conjugation matrix $\mathcal{C} = \begin{pmatrix} \mathcal{C}_+ & 0 \\ 0 & \mathcal{C}_- \end{pmatrix}$. In the correspondence (9), we use the symbolic notations $\omega_{[2p]}\Sigma^{(2p)} = \omega_{[2p]\,M_1 \cdots M_{2p}} \Sigma^{M_{2p} \cdots M_1}$. These relations which originate from the properties of the Dirac Γ–matrices, translate in fact (6) with explicit normalizations. Conversely, using the trace properties of the Γ matrices one gets

$$\begin{aligned}
\omega_{[2p]} &= \frac{1}{2^{2n}} \mathrm{Tr}\left(\Sigma^{(2p)} \Psi_{++}\right) , \quad \omega_{[2n]_+} = \frac{1}{2^{2n}} \mathrm{Tr}\left(\Sigma^{(2n)} \Psi_{++}\right) , \\
{\omega'}_{[2p]} &= \frac{1}{2^{2n}} \mathrm{Tr}\left(\tilde{\Sigma}^{(2p)} \Psi_{--}\right) , \quad \omega'_{[2n]_-} = \frac{1}{2^{2n}} \mathrm{Tr}\left(\tilde{\Sigma}^{(2n)} \Psi_{--}\right) , \\
\omega_{[2p+1]} &= \frac{1}{2^{2n}} \mathrm{Tr}\left(\tilde{\Sigma}^{(2p+1)} \Psi_{+-}\right) , \\
{\omega'}_{[2p+1]} &= \frac{1}{2^{2n}} \mathrm{Tr}\left(\Sigma^{(2p+1)} \Psi_{-+}\right) .
\end{aligned} \tag{10}$$

In components this gives for instance ${\omega_{[2p]}}^{M_1 \dots M_{2p}} = 2^{-2n} \mathrm{Tr}\left(\Sigma^{M_1 \dots M_{2p}} \Psi_{++}\right)$. Using the relations (9) and (10) can associate the p–forms listed in (7) to the multiplets $\Xi_{\pm\pm}$. We now calculate the transformation laws of these various p–forms. For example, for Ξ_{++}, from the transformation

$$\delta_\varepsilon \Xi_{++} = \left(\varepsilon^M V_M \boldsymbol{\Psi}_+\right) \otimes \boldsymbol{\Omega}_+ \tag{11}$$

we obtain

$$\begin{aligned}
\delta_\varepsilon \Xi_{1++} &= \varepsilon^M \Sigma_M \bar{\Xi}_{2-+} , \\
\delta_\varepsilon \bar{\Xi}_{2-+} &= \varepsilon^M \tilde{\Sigma}_M \Xi_{3++} , \\
\delta_\varepsilon \Xi_{3++} &= \varepsilon^M \partial_M \Xi_{1++} .
\end{aligned} \tag{12}$$

To calculate from (12) the transformation laws of the p–forms, we use the identities

$$\begin{aligned}
\Sigma_{M_1 \dots M_{2k}} \tilde{\Sigma}_{M_{2k+1}} &= \Sigma_{M_1 \dots M_{2k} M_{2k+1}} + \eta_{M_{2k} M_{2k+1}} \Sigma_{M_1 \dots M_{2k-1}} + \text{perm.} \\
\Sigma_{M_1 \dots M_{2k+1}} \Sigma_{M_{2k+2}} &= \Sigma_{M_1 \dots M_{2k+1} M_{2k+2}} + \eta_{M_{2k+1} M_{2k+2}} \Sigma_{M_1 \dots M_{2k}} + \text{perm.}
\end{aligned} \tag{13}$$

(where perm means sum on all permutations with the sign corresponding to its signature) for $k<n$ (when $k\geq n$ the situation is more involved because the $\Sigma^{(2k+2)}$, $\tilde{\Sigma}^{(2k+1)}$ matrices are related to the $\Sigma^{(2k+2-2n)}$, $\tilde{\Sigma}^{(2k+1-2n)}$ matrices). Similar relations hold for the $\tilde{\Sigma}^{(k)}$ matrices.

To obtain from (12) the transformation properties of the $p-$forms, we proceed differently for Ξ_{1++} and $\bar{\Xi}_{2-+}$, in order to avoid the presence of $\Sigma^{(\ell)}$ or $\tilde{\Sigma}^{(\ell)}$ matrices with $\ell>2n$.

Starting from $\delta_\varepsilon\bar{\Xi}_{2-+}=\varepsilon^M\tilde{\Sigma}_M\Xi_{3++}$, and using (10) we first have $\delta_\varepsilon\tilde{A}_{[2p+1]}=\varepsilon^M 2^{-2n}\mathrm{Tr}\left(\Sigma^{(2p+1)}\tilde{\Sigma}_M\Xi_{3++}\right)$. Using (13), we calculate $\Sigma^{(2p+1)}\tilde{\Sigma}_M$. Then, from the trace identities for the Σ matrices, one obtains the transformations laws for $\tilde{A}_{[2p+1]}$ (see (14)).

In order to calculate $\delta_\varepsilon\Xi_{1++}=\varepsilon^M\Sigma_M\bar{\Xi}_{2-+}$, we proceed in the reverse order. Firstly, using (9) and the identity (13), we compute the product $\Sigma_M\bar{\Xi}_{2-+}$. Then, using the trace formulae, we get the transformation laws of $A_{[2p]}$.

The last case $\delta_\varepsilon\Xi_{3++}$ is not difficult to handle. Finally we get

$$
\begin{array}{llll}
\delta_\varepsilon A_{[0]} & = i_\varepsilon\tilde{A}_{[1]} & \delta_\varepsilon\tilde{A}_{[1]} & = i_\varepsilon\tilde{\tilde{A}}_{[2]}+\tilde{\tilde{A}}_{[0]}\wedge\varepsilon \\
 & \vdots & & \vdots \\
\delta_\varepsilon A_{[2p]} & = i_\varepsilon\tilde{A}_{[2p+1]}+\tilde{A}_{[2p-1]}\wedge\varepsilon & \delta_\varepsilon\tilde{A}_{[2p+1]} & = i_\varepsilon\tilde{\tilde{A}}_{[2p+2]}+\tilde{\tilde{A}}_{[2p]}\wedge\varepsilon \\
 & \vdots & & \vdots \\
\delta_\varepsilon A_{[2n]_+} & = \tilde{A}_{[2n-1]}\wedge\varepsilon - i\,{}^\star\!\left(\tilde{A}_{[2n-1]}\wedge\varepsilon\right) & \delta_\varepsilon\tilde{A}_{[2n-1]} & = i_\varepsilon\tilde{\tilde{A}}_{[2n]_+}+\tilde{\tilde{A}}_{[2n-2]}\wedge\varepsilon
\end{array}
\tag{14}
$$

$$
\delta_\varepsilon\tilde{\tilde{A}}_{[0]}=\varepsilon A_{[0]},\qquad\ldots\qquad\delta_\varepsilon\tilde{\tilde{A}}_{[2n]_+}=\varepsilon A_{[2n]_+}.
$$

In these relations $i_\varepsilon A$ represents the inner product of ε and A[2]), $A\wedge\varepsilon$ represents the exterior product of A and ε, εA represents the action of the vector field $\varepsilon=\varepsilon^M\partial_M$ on A and ${}^\star A$ represents the dual of A[3]). The terms $-i\,{}^\star\!\left(\tilde{A}_{[2n-1]}\wedge\varepsilon\right)$ in $\delta_\varepsilon A_{[2n]_+}$ preserve the self–dual character of $A_{[2n]_+}$. Similar transformation laws can be obtained for the other multiplets. In the case of multiplets involving anti–self–dual $2n$–forms, the $-i$ becomes a $+i$, in perfect agreement with the complex conjugation (8).

3 Invariant Lagrangians

The transformations (14) suggest that an invariant Lagrangian should contain only zero–graded terms. Thus, one is forced to couple the fields in the (-1)–graded sector to the fields in the 1–graded sector and the fields in the 0–graded sector to themselves. Furthermore, if we consider for example the Ξ_{++} multiplet, in order

[2]) In this paper, we take $(i_\varepsilon A_{[p]})_{M_1\cdots M_{p-1}}=A_{[p]\,M_1\cdots M_p}\varepsilon^{M_p}$.

[3]) We remark that the transformation laws (14) have a geometrical interpretation in terms of inner and exterior product.

to have a real Lagrangian, one has also to take into consideration the conjugate multiplet Ξ_{--} (see (8)). For the Ξ_{++} and Ξ_{--} multiplets, the Lagrangian writes

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}(\Xi_{++}) + \mathcal{L}(\Xi_{--}) = \mathcal{L}_{[0]} + \ldots + \mathcal{L}_{[2n]} + \mathcal{L}'_{[0]} \quad + \ldots + \mathcal{L}'_{[2n]} = \\
&= \mathrm{d}A_{[0]} \mathrm{d}\tilde{\tilde{A}}_{[0]} + \ldots - \\
&\quad -\frac{1}{2}\frac{1}{(2p+2)!}\,\mathrm{d}\tilde{A}_{[2p+1]}\,\mathrm{d}\tilde{A}_{[2p+1]} - \frac{1}{2}\frac{1}{(2p)!}\,\mathrm{d}^\dagger\tilde{A}_{[2p+1]}\,\mathrm{d}^\dagger\tilde{A}_{[2p+1]} + \\
&\quad +\frac{1}{(2p+3)!}\,\mathrm{d}A_{[2p+2]}\,\mathrm{d}\tilde{\tilde{A}}_{[2p+3]} + \frac{1}{(2p+1)!}\,\mathrm{d}^\dagger A_{[2p+2]}\,\mathrm{d}^\dagger\tilde{\tilde{A}}_{[2p+2]} + \\
&\quad + \ldots + \\
&\quad +\frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}A_{[2n]_+}\,\mathrm{d}\tilde{\tilde{A}}_{[2n]_+} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger A_{[2n]_+}\,\mathrm{d}^\dagger\tilde{\tilde{A}}_{[2n]_+} + \\
&\quad +\mathrm{d}A'_{[0]}\,\mathrm{d}\tilde{\tilde{A}}'_{[0]} + \ldots - \qquad\qquad\qquad\qquad (15)\\
&\quad -\frac{1}{2}\frac{1}{(2p+2)!}\,\mathrm{d}\tilde{A}'_{[2p+1]}\,\mathrm{d}\tilde{A}'_{[2p+1]} - \frac{1}{2}\frac{1}{(2p)!}\,\mathrm{d}^\dagger\tilde{A}'_{[2p+1]}\,\mathrm{d}^\dagger\tilde{A}'_{[2p+1]} + \\
&\quad +\frac{1}{(2p+3)!}\,\mathrm{d}A'_{[2p+2]}\,\mathrm{d}\tilde{\tilde{A}}'_{[2p+3]} + \frac{1}{(2p+1)!}\,\mathrm{d}^\dagger A'_{[2p+2]}\,\mathrm{d}^\dagger\tilde{\tilde{A}}'_{[2p+2]} + \\
&\quad + \ldots + \\
&\quad +\frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}A'_{[2n]_-}\,\mathrm{d}\tilde{\tilde{A}}'_{[2n]_-} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger A'_{[2n]_-}\,\mathrm{d}^\dagger\tilde{\tilde{A}}'_{[2n]_-}\,.
\end{aligned}
$$

Here $\omega_{[p]}\omega'_{[p]}$ stands for $\omega_{[p]\,M_1\ldots M_p}\ \omega'_{[p]}{}^{M_1\ldots M_P}$, where $\omega_{[p]}$ and $\omega'_{[p]}$ are two p–forms[4]). In the Lagrangian (15), $\mathrm{d}A_{[p]}$ is the exterior derivative of $A_{[p]}$ and $\mathrm{d}^\dagger A_{[p]}$ its adjoint $\mathrm{d}^\dagger A_{[p]} = (-1)^{pD+D\star}\mathrm{d}^\star A_{[p]} = {}^\star\mathrm{d}^\star A_{[p]}$ for even D.

To prove that (15) is invariant under (14), we firstly note that $\delta_\varepsilon\mathcal{L}(\Xi_{++})$ and $\delta_\varepsilon\mathcal{L}(\Xi_{--})$ do not mix. It is thus sufficient to check separately their invariance, which we do here only for $\mathcal{L}(\Xi_{++})$ as an illustration. Starting from a specific normalization for $\mathcal{L}_{[0]}$, its variation fixes the normalization for $\mathcal{L}_{[1]}$. By a step-by-step process, the normalizations for $\mathcal{L}_{[p]}, 0 \le p \le 2n$ are also fixed. At the very end, all the terms of $\delta_\varepsilon\mathcal{L}$ compensate each others, up to a total derivative. Thus, the Lagrangian (15) is invariant.

If one considers the terms involving the (anti–)self–dual $2n$–form one can have further simplifications. Indeed, for the self–dual $2n$–form we have

$$
\begin{aligned}
\mathcal{L}_{[2n]} &= \frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}A_{[2n]_+}\,\mathrm{d}\tilde{\tilde{A}}_{[2n]_+} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger A_{[2n]_+}\,\mathrm{d}^\dagger\tilde{\tilde{A}}_{[2n]_+} = \\
&= \frac{1}{(2n+1)!}\,\mathrm{d}A_{[2n]_+}\,\mathrm{d}\tilde{\tilde{A}}_{[2n]_+}
\end{aligned}
$$

because of the self–duality condition ${}^\star A_{[2n]_+} = \mathrm{i}A_{[2n]_+}$. A more interesting way of

[4]) In p–form notation we can rewrite $\int \mathrm{d}^D x\ \omega_{[p]}\omega'_{[p]}$ as $\int \omega_{[p]} \wedge {}^\star\omega'_{[p]}$.

regrouping the terms involving the self–dual and the anti–self–dual $2n$–forms is

$$
\begin{aligned}
\mathcal{L}_{[2n]} + \mathcal{L}'_{[2n]} =& \frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}A_{[2n]_+}\,\mathrm{d}\tilde{\tilde{A}}_{[2n]_+} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger A_{[2n]_+}\,\mathrm{d}^\dagger\tilde{\tilde{A}}_{[2n]_+} + \\
&+\frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}A'_{[2n]_-}\,\mathrm{d}\tilde{\tilde{A}}'_{[2n]_-} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger A'_{[2n]_-}\,\mathrm{d}^\dagger\tilde{\tilde{A}}'_{[2n]_-} = \\
=& \frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}\big(A_{[2n]_+} + A'_{[2n]_-}\big)\,\mathrm{d}\big(\tilde{\tilde{A}}_{[2n]_+} + \tilde{\tilde{A}}'_{[2n]_-}\big) + \\
&+\frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger\big(A_{[2n]_+} + A'_{[2n]_-}\big)\mathrm{d}^\dagger\big(\tilde{\tilde{A}}_{[2n]_+} + \tilde{\tilde{A}}'_{[2n]_-}\big),
\end{aligned}
\tag{16}
$$

since $A_{[2n]_+}A'_{[2n]_-} = 0$ when $D = 1 + (4n-1)$, for a self–dual and an anti–self–dual $2n$–forms. The final real $2n$–forms

$$
A_{1\,[2n]} = \frac{1}{\sqrt{2}}\left(A_{[2n]_+} + A'_{[2n]_-}\right), \tilde{\tilde{A}}_{1\,[2n]} = \frac{1}{\sqrt{2}}\left(\tilde{\tilde{A}}_{[2n]_+} + \tilde{\tilde{A}}'_{[2n]_-}\right), \tag{17}
$$

are neither self–dual nor anti–self–dual, which is in agreement with the representation theory of the Poincaré algebra.

Considering the gauge invariance

$$
A_{[p]} \to A_{[p]} + d\chi_{[p-1]} \quad \text{with } p \geq 1\,, \tag{18}
$$

(where $\chi_{[p-1]}$ is a $(p-1)$–form), the terms involving $\mathrm{d}^\dagger$ in $\mathcal{L}$ (15) fix partially the gauge

$$
\mathrm{d}^\dagger\,\mathrm{d}\chi_{[p-1]} = 0\,. \tag{19}
$$

Hence, these terms involving the $d^\dagger$ operators can be seen as gauge fixing terms (Feynman gauge adapted for p–forms). Another way of seeing this is to rewrite the Lagrangian using Fermi–like terms. For instance, for the $\mathcal{L}_{[2p+1]}$ part in (15) we have

$$
\begin{aligned}
\mathcal{L}_{[2p+1]} =& -\frac{1}{2}\left(\frac{1}{(2p+2)!}\,\mathrm{d}\tilde{A}_{[2p+1]}\,\mathrm{d}\tilde{A}_{[2p+1]} + \frac{1}{(2p)!}\,\mathrm{d}^\dagger\tilde{A}_{[2p+1]}\,\mathrm{d}^\dagger\tilde{A}_{[2p+1]}\right) = \\
=& -\frac{1}{2}\frac{1}{(2p+1)!}\,\partial_{M_1}\tilde{A}_{[2p+1]\,M_2\cdots M_{2p+2}}\partial^{M_1}\tilde{A}_{[2p+1]}{}^{M_2\cdots M_{2p+2}} = \\
=& -\frac{1}{2}\frac{1}{(2p+1)!}\,\partial\tilde{A}_{[2p+1]}\partial\tilde{A}_{[2p+1]}\,.
\end{aligned}
$$

We now look at the physical degrees of freedom in (15). By construction, the various p–forms are complex; thus we introduce real p–forms through

$$
\begin{aligned}
A_{1\,[2p]} &= \tfrac{1}{\sqrt{2}}\left(A_{[2p]} + A'_{[2p]}\right), & A_{2\,[2p]} &= \tfrac{\mathrm{i}}{\sqrt{2}}\left(A_{[2p]} - A'_{[2p]}\right), \\
\tilde{A}_{1\,[2p+1]} &= \tfrac{1}{\sqrt{2}}\left(\tilde{A}_{[2p+1]} + \tilde{A}'_{[2p+1]}\right), & \tilde{A}_{2\,[2p+1]} &= \tfrac{\mathrm{i}}{\sqrt{2}}\left(\tilde{A}_{[2p+1]} - \tilde{A}'_{[2p+1]}\right), \\
\tilde{\tilde{A}}_{1\,[2p]} &= \tfrac{1}{\sqrt{2}}\left(\tilde{\tilde{A}}_{[2p]} + \tilde{\tilde{A}}'_{[2p]}\right), & \tilde{\tilde{A}}_{2\,[2p]} &= \tfrac{\mathrm{i}}{\sqrt{2}}\left(\tilde{\tilde{A}}_{[2p]} - \tilde{\tilde{A}}'_{[2p]}\right)
\end{aligned}
\tag{20}
$$

with $p = 0, \ldots, n-1$. Observe that in the substitution (17) we have two $2n$–forms, although for the other substitution (20) we have four p–forms. After these redefinitions, we have a Lagrangian expressed only with real fields. However, some of the terms in $\mathcal{L}$ are not diagonal, namely the ones involving the fields of the (-1)– and 1–graded sectors ($e.g$ $\frac{1}{(2p+1)!}\,\mathrm{d}A_{1\,[2p]}\,\mathrm{d}\tilde{\tilde{A}}_{1\,[2p]} + \frac{1}{(2p-1)!}\,\mathrm{d}^\dagger A_{1\,[2p]}\,\mathrm{d}^\dagger \tilde{\tilde{A}}_{1\,[2p]}$). Therefore, in order to diagonalise $\mathcal{L}$ we introduce

$$\begin{array}{lll}
\hat{A}_{1\,[2p]} = \frac{1}{\sqrt{2}}\left(A_{1\,[2p]} + \tilde{\tilde{A}}_{1\,[2p]}\right), & \hat{\hat{A}}_{1\,[2p]} = \frac{1}{\sqrt{2}}\left(A_{1\,[2p]} - \tilde{\tilde{A}}_{1\,[2p]}\right), & p = 0, \ldots, n\,, \\
\hat{A}_{2\,[2p]} = \frac{1}{\sqrt{2}}\left(A_{2\,[2p]} + \tilde{\tilde{A}}_{2\,[2p]}\right), & \hat{\hat{A}}_{2\,[2p]} = \frac{1}{\sqrt{2}}\left(A_{2\,[2p]} - \tilde{\tilde{A}}_{2\,[2p]}\right), & p = 0, \ldots, n-1\,,
\end{array} \tag{21}$$

which are mixtures of fields of gradation (-1) and 1.

Finally, after the field redefinitions (20) and (21), the new fields are $\hat{A}_{1\,[2p]}$, $\hat{A}_{2\,[2p]}$, $\hat{\hat{A}}_{1\,[2p]}$, $\hat{\hat{A}}_{2\,[2p]}$, $\tilde{A}_{1\,[2p+1]}$, $\tilde{A}_{2\,[2p+1]}$, $p = 0, \ldots, n-1$ and $\hat{A}_{1\,[2n]}$, $\hat{\hat{A}}_{1\,[2n]}$. Usually, the kinetic term for a p–form $\omega_{[p]}$ writes, with our convention for the metric, $(-1)^p \frac{1}{2(p+1)!}\,\mathrm{d}\omega_{[p]}\,\mathrm{d}\omega_{[p]}$. Expressed with these fields, the Lagrangian has these conventional kinetic terms except that (i) we have also gauge fixing terms in addition to the kinetic terms and (ii) some of the fields ($\hat{\hat{A}}_{1\,[2p]}$, $\hat{A}_{2\,[2p]}$, $\tilde{A}_{2\,[2p+1]}$, $p = 0, \ldots, n-1$, $\hat{\hat{A}}_{1\,[2n]}$), have the opposite sign for the kinetic and the gauge fixing terms. This implies that these fields have an energy density not bounded from bellow.

One possible way to avoid this problem is based on Hodge duality. For a given p–form $\omega_{[p]}$ we have

$$\begin{aligned}
&(-1)^p \left(\frac{1}{(p+1)!}\,\mathrm{d}\omega_{[p]}\,\mathrm{d}\omega_{[p]} + \frac{1}{(p-1)!}\,\mathrm{d}^\dagger\omega_{[p]}\,\mathrm{d}^\dagger\omega_{[p]}\right) = \\
&= (-1)^p \left(\frac{1}{(D-p-1)!}\,\mathrm{d}^\dagger\rho_{[D-p]}\,\mathrm{d}^\dagger\rho_{[D-p]} + \frac{1}{(D-p+1)!}\,\mathrm{d}\rho_{[D-p]}\,\mathrm{d}\rho_{[D-p]}\right)
\end{aligned} \tag{22}$$

with $\rho = {}^\star\omega$ the Hodge dual of ω. This substitution is due to the special form of our Lagrangian: through the substitution $\omega \to \rho$, the kinetic term of ω becomes the gauge fixing term of ρ and *vice versa*. Thus the following substitutions are made

$$\begin{aligned}
\hat{\hat{A}}_{1\,[2p]} &\to \hat{\hat{B}}_{1\,[D-2p]} = {}^\star\hat{\hat{A}}_{1\,[2p]}\,, \\
\hat{A}_{2\,[2p]} &\to \hat{B}_{2\,[D-2p]} = {}^\star\hat{A}_{2\,[2p]}\,, \\
\tilde{A}_{2\,[2p+1]} &\to \tilde{B}_{2\,[D-2p-1]} = {}^\star\tilde{A}_{2\,[2p+1]}\,, \\
\hat{\hat{A}}_{1\,[2n]} &\to \hat{\hat{B}}_{1\,[D-2n]} = {}^\star\hat{\hat{A}}_{1\,[2n]}
\end{aligned} \tag{23}$$

with $p = 0, \ldots, n-1$. Let us emphasize that the substitutions (23) are done with respect to the gauge fields. This is quite different from the usual duality transformations (generalizing the electric–magnetic duality) where the duality transformations are done with respect to the field strengths. With the "duality" transformations (23) the number of degree of freedom is not the same for $\omega_{[p]}$ and $\rho_{[D-p]} = {}^\star\omega_{[p]}$, which is not the case for the usual duality transformations. Hence, the two Lagrangians (15) and (24) (see below) describe inequivalent theories.

Thus, starting from the Lagrangian (15) and performing the field redefinitions (20), (21) and (23) we get

$$
\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}\,\mathrm{d}\hat{A}_{1\,[0]}\,\mathrm{d}\hat{A}_{1\,[0]} + \frac{1}{2}\,\mathrm{d}\hat{\hat{A}}_{2\,[0]}\,\mathrm{d}\hat{\hat{A}}_{2\,[0]} + \\
& + \ldots - \\
& - \frac{1}{2}\frac{1}{(2p+2)!}\,\mathrm{d}\tilde{A}_{1\,[2p+1]}\,\mathrm{d}\tilde{A}_{1\,[2p+1]} - \frac{1}{2}\frac{1}{(2p)!}\,\mathrm{d}^\dagger\tilde{A}_{1\,[2p+1]}\,\mathrm{d}^\dagger\tilde{A}_{1\,[2p+1]} + \\
& + \frac{1}{2}\frac{1}{(2p+3)!}\,\mathrm{d}\hat{A}_{1\,[2p+2]}\,\mathrm{d}\hat{A}_{1\,[2p+2]} + \frac{1}{2}\frac{1}{(2p+1)!}\,\mathrm{d}^\dagger\hat{A}_{1\,[2p+2]}\,\mathrm{d}^\dagger\hat{A}_{1\,[2p+2]} + \\
& + \frac{1}{2}\frac{1}{(2p+3)!}\,\mathrm{d}\hat{\hat{A}}_{2\,[2p+2]}\,\mathrm{d}\hat{\hat{A}}_{2\,[2p+2]} + \frac{1}{2}\frac{1}{(2p+1)!}\,\mathrm{d}^\dagger\hat{\hat{A}}_{2\,[2p+2]}\,\mathrm{d}^\dagger\hat{\hat{A}}_{2\,[2p+2]} + \\
& + \ldots + \\
& + \frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}\hat{A}_{1\,[2n]}\,\mathrm{d}\hat{A}_{1\,[2n]} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger\hat{A}_{1\,[2n-1]}\,\mathrm{d}^\dagger\hat{A}_{1\,[2n]} + \\
& + \frac{1}{2}\frac{1}{(2n+1)!}\,\mathrm{d}\hat{\hat{B}}_{1\,[2n]}\,\mathrm{d}\hat{\hat{B}}_{1\,[2n]} + \frac{1}{2}\frac{1}{(2n-1)!}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[2n-1]}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[2n]} + \\
& + \ldots + \\
& + \frac{1}{2}\frac{1}{(D-2p-1)!}\,\mathrm{d}\hat{\hat{B}}_{1\,[D-2p-2]}\,\mathrm{d}\hat{\hat{B}}_{1\,[D-2p-2]} + \\
& + \frac{1}{2}\frac{1}{(D-2p-3)!}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[D-2p-2]}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[D-2p-2]} + \\
& + \frac{1}{2}\frac{1}{(D-2p-1)!}\,\mathrm{d}\hat{B}_{2\,[D-2p-2]}\,\mathrm{d}\hat{B}_{2\,[D-2p-2]} + \\
& + \frac{1}{2}\frac{1}{(D-2p-3)!}\,\mathrm{d}^\dagger\hat{B}_{2\,[D-2p-2]}\,\mathrm{d}^\dagger\hat{B}_{2\,[D-2p-2]} - \\
& - \frac{1}{2}\frac{1}{(D-2p)!}\,\mathrm{d}\tilde{B}_{2\,[D-2p-1]}\,\mathrm{d}\tilde{B}_{2\,[D-2p-1]} - \\
& - \frac{1}{2}\frac{1}{(D-2p-2)!}\,\mathrm{d}^\dagger\tilde{B}_{2\,[D-2p-1]}\,\mathrm{d}^\dagger\tilde{B}_{2\,[D-2p-1]} + \\
& + \ldots + \\
& + \frac{1}{2}\frac{1}{(D-1)!}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[D]}\,\mathrm{d}^\dagger\hat{\hat{B}}_{1\,[D]} + \frac{1}{2}\frac{1}{(D-1)!}\,\mathrm{d}^\dagger\hat{B}_{2\,[D]}\,\mathrm{d}^\dagger\hat{B}_{2\,[D]}.
\end{aligned}
\tag{24}
$$

At the very end, we have one one–form, one three–form, ..., one $(D-1)$–form in the zero–graded sector and two zero–forms, two two–forms, ... and two D–form in the mixture of the sectors of gradation (-1) and 1. In the Lagrangian (24), all the p–forms have a kinetic term and a gauge fixing term; the only exceptions are the zero–forms, which have only kinetic terms, and the D–forms, which have only gauge fixing terms.

The gauge invariance (18) and the field equations imply (for a p–form $A_{[p]}$, with $p \le D-2$) $P^M A_{[p]\,M M_2 \cdots M_p} = 0$ and $P^2 = 0$ (with P^M the momentum), thus

$A_{[p]}$ gives rise to a massless state in the p–order antisymmetric representation of the little group $SO(D-2)$. But, in our decomposition, there are also appearing $p-$forms with $p = D-1, D$. Of course these p–forms do not propagate. It is interesting to note that such p–forms also appear in $D = 10$ in the context of type IIA, IIB string theory [9].

As we have seen, cubic symmetry is compatible with gauge invariance if the gauge is partially fixed. Thus, in order that the field equations obtained from (24) reproduce the usual field equations for a free p–form, one has to impose the analogous of the Lorentz condition $d^\dagger A_{[p]} = 0$ (to eliminate the unphysical components). All this has to be implemented at the quantum level, for instance using an appropriate extension of the Gupta–Bleuler quantization for p–forms. Of course one has to check if this procedure is compatible with cubic symmetry.

Finally, a massless p–form ($p < D-1$) has $\begin{pmatrix} p \\ D-1 \end{pmatrix}$ degrees of freedom off–shell and $\begin{pmatrix} p \\ D-2 \end{pmatrix}$ on–shell, a $(D-1)$–form has 1 degree of freedom off–shell and no degree of freedom on–shell and a D–form has no degree of freedom off– and on–shell. Therefore, one can check that we have twice as many degrees of freedom in the (-1)– and 1–graded sector than in the 0–graded sector.

To the Lagrangian (15) one can add the following invariant mass terms

$$\begin{aligned}\mathcal{L}_m = m^2 \Big(& A_{[0]}\tilde{\tilde{A}}_{[0]} + \ldots - \frac{1}{2}\frac{1}{(2p+1)!}\tilde{A}_{[2p+1]}\tilde{A}_{[2p+1]} + \frac{1}{(2p+2)!}A_{[2p+2]}\tilde{\tilde{A}}_{[2p+2]} + \\ & + \ldots + \frac{1}{(2n)!}A_{[2n]}\tilde{\tilde{A}}_{[2n]} + A'_{[0]}\tilde{\tilde{A}}'_{[0]} + \ldots - \frac{1}{2}\frac{1}{(2p+1)!}\tilde{A}'_{[2p+1]}\tilde{A}'_{[2p+1} + \\ & + \frac{1}{(2p+2)!}A'_{[2p+2]}\tilde{\tilde{A}}'_{[2p+2]} + \ldots + \frac{1}{(2n)!}A'_{[2n]}\tilde{\tilde{A}}'_{[2n]}\Big). \end{aligned} \tag{25}$$

This means that with the same multiplets (7) one can have massless or massive invariant terms. Thus, the matrix representations (2) and (3) do not depend on whether or not we are considering massless or massive particles. Of course one can also proceed along the same lines as before (Eq.(20)−(21)−(23)), in order to express $\mathcal{L}_m$ in terms of the physical degrees of freedom.

4 Conclusion

In this paper, we have explicitly built a D–dimensional field theory implementation of the cubic symmetry algebra. This symmetry acts naturally on p–forms and has a geometrical interpretation in terms of inner and exterior products with a one–form, the parameter of the transformation. Due to the interest for p–forms, mostly within supergravity and superstring theories, it is appealing to consider new possible symmetries on these antisymmetric tensor fields. As we have seen, the cubic symmetry invariance requires the presence of gauge fixing terms, thus placing the theory in a Feynman gauge adapted for p–forms. One of the possible invariant Lagrangians contains p–forms, with $0 \le p \le D$. Amongst them, the $(D-1)$– and D–forms are non-propagating, which is also the case for types IIA and IIB string theories [9]. Finally, when one considers the way the p–forms $A_{[2p]}$, $\tilde{A}_{[2p+1]}$ transform under cubic symmetry (14), one sees an analogy with the T–duality transformations relating the p–forms of types IIA and type IIB strings. Indeed, looking at (14) and taking ε along a given direction, the transformation laws correspond to the T–duality formulae (C.1) and (C.2) of Ref.[2] (when gravity is not present).

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