Method of categorical extension of Cayley–Klein groups *)

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The method of categorical extension of the Cayley–Klein groups is developed. The method uses the Cayley–Klein spaces, as objects of the Cayley–Klein category, endowed with all possible linear relations or bilinear forms as morphisms.

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1 Introduction

In this work we undertake an attempt to formulate the method of categorical extension of the theory of a Cayley–Klein group $G(\mathbf{j})$ [1 – 3].
Let us define *Pimenov algebra* $\mathcal{D}_n(\boldsymbol{\iota};\mathbb{C})$ as an associative algebra with unit over complex number field and with nilpotent commutative generators $\iota_k,\ \iota_k^2=0,\ \iota_k\iota_m=\iota_m\iota_k\neq 0$, $k\neq m,\ \ k,m=1,\ldots,n$. The general element of $\mathcal{D}_n(\boldsymbol{\iota};\mathbb{C})$ is in the form

$$d=d_0+\sum_{p=1}^{n}\sum_{k_1<\ldots<k_p} d_{k_1\ldots k_p}\iota_{k_1}\ldots\iota_{k_p},\quad d_0,d_{k_1\ldots k_p}\in\mathbb{C}.$$

For $n=1$ we have $\mathcal{D}_1(\iota_1;\mathbb{C})\ni d=d_0+d_1\iota_1$, i.e. the elements d are dual (or Study) numbers when $d_0,d_1\in\mathbb{R}$. For $n=2$ the general element of $\mathcal{D}_2(\iota_1,\iota_2;\mathbb{C})$ is $d=d_0+d_1\iota_1+d_2\iota_2+d_{12}\iota_1\iota_2$. Two elements $d,\tilde{d}\in\mathcal{D}_n(\boldsymbol{\iota};\mathbb{C})$ are equal if and only if $d_0=\tilde{d}_0,\ d_{k_1\ldots k_p}=\tilde{d}_{k_1\ldots k_p},\ p=1,\ldots,n$. If $d=d_k\iota_k$ and $\tilde{d}=\tilde{d}_k\iota_k$, then the condition $d=\tilde{d}$, which is equivalent to $d_k\iota_k=\tilde{d}_k\iota_k$, make possible the consistently definition of the division of nilpotent generator ι_k by itself, namely: $\iota_k/\iota_k=1,\ k=1,\ldots,n$. Let us stress that the division of different nilpotent generators $\iota_k/\iota_p,\ k\neq p$, as well as the division of complex number by nilpotent generators $a/\iota_k,\ a\in\mathbb{C}$ are not defined. It is convenient to regard the Cayley–Klein algebras $\mathcal{CK}_n(\mathbf{j};\mathbb{K})$, where $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ and the parameters $j_k=1,\iota_k,i;\ k=1,\ldots,n$. If m parameters are nilpotent $j_{k_s}=\iota_s,\ s=1,\ldots,m$ and the other are equal to unit, then we have Pimenov algebra $\mathcal{D}_m(\boldsymbol{\iota};\mathbb{C})$.

The matrix realization allows a natural interpretation of the Cayley–Klein algebras as the Lie algebras of the motion groups of the homogeneous symmetric spaces with a Hermitian metric. Let us consider the space $V^{N+1}(\mathbf{j})$ endowed with a Hermitian (sesqui)linear form $\langle\,.\,|\,.\,\rangle_j : V^{N+1}(\mathbf{j})\otimes V^{N+1}(\mathbf{j})\to\mathbb{K}^{N+1}\otimes\mathbb{K}^{N+1}$ defined by

$$\langle\mathbf{a}|\mathbf{b}\rangle_{\mathbf{j}}:=\bar{a}^0b^0+\bar{a}^1j_1b^1+\bar{a}^2j_1j_2b^2+\ldots+\bar{a}^Nj_1\cdots j_Nb^N=\bar{a}^0b^0+\sum_{i=1}^{N}\bar{a}^i(\prod_{l=1}^{i}j_l)b^i\,,\quad(1)$$

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where $\mathbf{a}, \mathbf{b} \in V^{N+1}(\mathbf{j})$ and $\bar{a}^i$ means the conjugation of the component a^i. For the moment, we assume that we are in the generic case with all $j_l \neq 0$. The underlying metric is provided by the matrix

$$\mathcal{I}_{\mathbf{j}} = \text{diag}\,(1,\, j_1,\, j_1 j_2, \ldots,\, j_1 \cdots j_N) \tag{2}$$

and the Cayley–Klein group $G(N+1, j_1, \ldots, j_N) \equiv G(N+1, \mathbf{j})$ is defined as the group of linear isometries of this Hermitian metric over space $V^{N+1}(\mathbf{j})$. Thus the isometry condition for an element U of the Cayley–Klein group $G(N+1, \mathbf{j})$

$$\langle U\mathbf{a}|U\mathbf{b}\rangle_j = \langle \mathbf{a}|\mathbf{b}\rangle_j \quad \forall\, \mathbf{a}, \mathbf{b} \in V^{N+1}(\mathbf{j})\,, \tag{3}$$

leads to the following relation

$$U^{\dagger}\mathcal{I}_{\mathbf{j}}U = \mathcal{I}_{\mathbf{j}} \quad \forall U \in G(N+1, \mathbf{j})\,, \tag{4}$$

which for the Cayley–Klein algebra implies

$$X^{\dagger}\mathcal{I}_{\mathbf{j}} + \mathcal{I}_{\mathbf{j}}X = 0 \quad \forall X \in \mathcal{CK}_{N+1}(\mathbf{j})\,. \tag{5}$$

Let $G(\mathbf{j})$ be a Cayley–Klein group. Then there exists a certain category $CK\mathbf{K}$ such that the group itself is the automorphism group of a certain object $V(\mathbf{j})$, while the semigroup $\Gamma(\mathbf{j})$ is the semigroup of endomorphisms of this same object. Furthermore, each representation $\rho(\mathbf{j})$ of Cayley–Klein group $G'(\mathbf{j})$ on a Cayley–Klein space $H(\mathbf{j})$ can be extended to a representation of the Cayley–Klein category $CK\mathbf{K}$. In other words, for each objects $W(\mathbf{j})$ of the Cayley–Klein category $CK\mathbf{K}$ we can construct a linear space $T(W(\mathbf{j}))$ and for each morphism $P(\mathbf{j}) : W(\mathbf{j}) \to W'(\mathbf{j})$ we can construct a linear operator $\tau(P(\mathbf{j})) : T(W(\mathbf{j})) \to T\,(W'(\mathbf{j}))$ such that for any morphisms $P(\mathbf{j}) : W(\mathbf{j}) \to W'(\mathbf{j})$ and $Q(\mathbf{j}) : W'(\mathbf{j}) \to W''(\mathbf{j})$ we have

$$\tau(Q(\mathbf{j})P(\mathbf{j})) = \tau(Q(\mathbf{j}))\tau(P(\mathbf{j}))\,, \tag{6}$$

with $T(V(\mathbf{j})) = H(\mathbf{j})$, and for all $g \in G$ the operators $\tau((g))$ and $\rho((g))$ are the same.

We note that all the spaces $T(W(\mathbf{j}))$ and all the operators $\tau(\mathbf{j})$ "grow out of" the one and only representation $\rho(\mathbf{j})$ of Cayley–Klein group $G(\mathbf{j})$ and the one and only space Cayley–Klein $H(\mathbf{j})$.

To define a Cayley–Klein category $CK\mathbf{K}$ we require the following data:

(a) a set $\mathrm{Ob}(CK\mathbf{K})$ of elements called the objects of the category $CK\mathbf{K}$;

(b) for any two objects $V(\mathbf{j}), W(\mathbf{j}) \in \mathrm{Ob}(CK\mathbf{K})$ a set $\mathrm{Mor}_{CK\mathbf{K}}(V(\mathbf{j}), W(\mathbf{j}))$ is defined, called the *morphisms* from $V(\mathbf{j})$ to $W(\mathbf{j})$ (when it is clear what the category in question is, we omit the index $CK\mathbf{K}$ and merely write $\mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$);

(c) for any $P(\mathbf{j})$ in $\mathrm{Mor}(V(\mathbf{j}), V'(\mathbf{j}))$ and $Q(\mathbf{j})$ in $\mathrm{Mor}(V'(\mathbf{j}), V''(\mathbf{j}))$ their product $Q(\mathbf{j})P(\mathbf{j})$ is defined in $\mathrm{Mor}(V(\mathbf{j}), V''(\mathbf{j}))$. The product must be associative: for any $P(\mathbf{j})$ in $\mathrm{Mor}(V(\mathbf{j}), V'(\mathbf{j}))$, $Q(\mathbf{j})$ in $\mathrm{Mor}(V'(\mathbf{j}), V''(\mathbf{j}))$, and $R(\mathbf{j})$ in $\mathrm{Mor}(V''(\mathbf{j}), V'''(\mathbf{j}))$, the formula

$$R(\mathbf{j})(Q(\mathbf{j})P(\mathbf{j})) = (R(\mathbf{j})Q(\mathbf{j}))P(\mathbf{j}) \tag{7}$$

holds;

(d) it is usually assumed that the set $\mathrm{Mor}(V(\mathbf{j}), V(\mathbf{j}))$ contains an element $1_{V(\mathbf{j})}$ called the *identity* such that, for any $P(\mathbf{j}) \in \mathrm{Mor}(Y(\mathbf{j}), V(\mathbf{j}))$, we have $P(\mathbf{j}) \cdot 1_{V(\mathbf{j})} = P(\mathbf{j})$ and, for any $Q(\mathbf{j}) \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$, we have $1_{V(\mathbf{j})} \cdot Q(\mathbf{j}) = Q(\mathbf{j})$.

To explain the idea of an analog of a structure on a category, let us consider first the category of linear relations. The objects of this category are linear spaces over a field $\mathbb{K}$ (and we will suppose them to be finite-dimensional). The morphisms $P(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ are the linear relations, that is, the subspaces $P(\mathbf{j})$ of $V(\mathbf{j}) \oplus W(\mathbf{j})$.

If $P(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ and $Q(\mathbf{j}) : W(\mathbf{j}) \rightrightarrows Y(\mathbf{j})$ are linear relations, then their product $Q(\mathbf{j})P(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows Y(\mathbf{j})$ is defined as follows: $(v(\mathbf{j}), y(\mathbf{j})) \in V(\mathbf{j}) \oplus Y(\mathbf{j})$ is contained in the subspace $Q(\mathbf{j})P(\mathbf{j})$ if there exists $\omega \in W(\mathbf{j})$ such that $(v(\mathbf{j}), \omega) \in P(\mathbf{j})$ and $(\omega, y(\mathbf{j})) \in Q(\mathbf{j})$ (and this is how one would want to define the product of "multivalued maps").

The following are defined for a linear relation $P(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ in the same way as for an operator:

(a) the *kernel* $\ker\ P(\mathbf{j})$ — the set of all $v(\mathbf{j}) \in V(\mathbf{j})$ such that $(v(\mathbf{j}), 0) \in P(\mathbf{j})$;

(b) the *image* $\mathrm{im}\, P(\mathbf{j})$ — the projection of $P(\mathbf{j})$ onto $W(\mathbf{j})$;

(c) the *domain of definition* $D(P(\mathbf{j}))$ — the projection of $P(\mathbf{j})$ onto $V(\mathbf{j})$.

(d) the *indefiniteness* $\mathrm{Indef}(P(\mathbf{j}))$; this is the set of $\omega \in W(\mathbf{j})$ such that $(0, \omega) \in P(\mathbf{j})$; if $P(\mathbf{j})$ is the graph of an operator then $\mathrm{Indef}(P(\mathbf{j})) = 0$;

(e) the *rank* $\mathrm{rk}(P(\mathbf{j}))$: $\mathrm{rk}(P(\mathbf{j})) = \dim D(P(\mathbf{j})) - \dim \ker P(\mathbf{j}) = \dim \mathrm{im} P(\mathbf{j}) - \dim \mathrm{Indef} P(\mathbf{j}) = \dim P(\mathbf{j}) - \dim \ker P(\mathbf{j}) - \dim \mathrm{Indef} P(\mathbf{j})$.

2 The real classical Cayley−Klein categories

All the real classical Cayley–Klein categories that can be obtained by the method of categorical extension are listed in Table 1.
The first column gives the name of the category. The second column gives the field (or division ring) $\mathbb{K}$, while the third gives the objects of the category $(p, q, n, \in \mathbb{Z})$; in all cases these objects are the spaces $V^{p,q}(\mathbf{j}) = V_+(\mathbf{j}) \oplus V_-(\mathbf{j}) = \mathbb{K}^p \oplus \mathbb{K}^q$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, consisting of the vectors $x = (x^+, x^-) = (x_0^+, x_1^+, \ldots, x_p^+; x_0^-, x_1^-, \ldots, x_q^-)$ and endowed with the Hermitian form (1). As well as the form Λ, which appears in the fourth column, and is introduced in $V(\mathbf{j})$ as a non-degenerate skew-symmetric, symmetric, Hermitian, or skew-Hermitian form $\Lambda = \Lambda_{V(\mathbf{j})}$ so that the spaces $V_+(\mathbf{j})$ and $V_-(\mathbf{j})$ are isotropic with respect to Λ. In Table 1 the field (or division ring) $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the type of form Λ are determined by the category $CK\mathbf{K}$. The morphisms $V(\mathbf{j}) \to W(\mathbf{j})$ are the linear relations $L(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ satisfying the two conditions:
(a) $L(\mathbf{j})$ contracts the form (1);
(b) $L(\mathbf{j})$ "preserves the form Λ", that is, form (1) is a maximal isotropic subspaces of $V(\mathbf{j}) \oplus W(\mathbf{j})$ with respect to the form

$$\Lambda_{V(\mathbf{j}) \oplus W(\mathbf{j})}((v, w), (v', w')) = \Lambda_{V(\mathbf{j})}(v, v') - \Lambda_{W(\mathbf{j})}(w, w') \,. \tag{8}$$

The fifth column gives the group $G(\mathbf{j}) = \mathrm{Aut}_A(V(\mathbf{j}))$ of automorphisms of $V(\mathbf{j})$.

Table 1. List of the real classical Cayley–Klein categories

Name	Field	Object	Form $\Lambda \equiv <,>$	$\mathrm{Aut}(V(\mathbf{j}))$
$CK\mathbf{K}$	$\mathbb{K}$	$V_+ \oplus V_-$		
$CK\mathbf{U}$	$\mathbb{C}$	$\mathbb{C}^p \oplus \mathbb{C}^q$		$U(p,q,\mathbf{j})$
$CK\mathbf{Sp}$	$\mathbb{C}$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$x_0^+ y_0^- - x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- - x_k^- y_k^+)$	$U(n,\mathbf{j})$
$CK\mathbf{SO}^*$	$\mathbb{C}$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$x_0^+ y_0^- + x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- + x_k^- y_k^+)$	$SO^*(2n,\mathbf{j})$
$CK\mathbf{GL}(\mathbb{C})$	$\mathbb{C}$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$x_0^+ y_0^- - x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- - x_k^- y_k^+)$	$GL(n,\mathbb{C},\mathbf{j})$
$CK\mathbf{O}(\mathbb{R})$	$\mathbb{R}$	$\mathbb{R}^p \oplus \mathbb{R}^q$		$O(p,q,\mathbf{j})$
$CK\mathbf{GL}(\mathbb{R})$	$\mathbb{R}$	$\mathbb{R}^p \oplus \mathbb{R}^q$	$x_0^+ y_0^- - x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- - x_k^- y_k^+)$	$GL(n,\mathbb{R},\mathbf{j})$
$CK\mathbf{O}(\mathbb{C})$	$\mathbb{R}$	$\mathbb{R}^n \oplus \mathbb{R}^n$	$x_0^+ y_0^- + x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- + x_k^- y_k^+)$	$O(n,\mathbb{C},\mathbf{j})$
$CK\mathbf{Sp}(\mathbb{H})$	$\mathbb{H}$	$\mathbb{H}^p \oplus \mathbb{H}^q$		$\mathrm{Sp}(p,q,\mathbf{j})$
$CK\mathbf{GL}(\mathbb{H})$	$\mathbb{H}$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$x_0^+ y_0^- - x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- - x_k^- y_k^+)$	$GL(n,\mathbb{H},\mathbf{j})$
$CK\mathbf{Sp}(\mathbb{C})$	$\mathbb{H}$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$x_0^+ y_0^- + x_0^- y_0^+ + \sum_{k=1} (\prod_{l=1}^{k} j_l)(x_k^+ y_k^- + x_k^- y_k^+)$	$\mathrm{Sp}(2n,\mathbb{C},\mathbf{j})$

3 The complex classical Cayley−Klein categories

The category $CK\mathbf{GA}$. The *objects* of $CK\mathbf{GA}$ are finite–dimensional complex linear Cayley–Klein spaces. The *morphisms* $\mathrm{Mor}_{CK\mathbf{GA}}(V(\mathbf{j}), W(\mathbf{j}))$ consist of all possible linear relations $V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$, together with the formal element $\mathrm{null} = \mathrm{null}_{V(\mathbf{j}),W(\mathbf{j})}$, which is not identified with any linear relations. Let $P(\mathbf{j}) \in \mathrm{Mor}_{CK\mathbf{GA}}(V(\mathbf{j}), W(\mathbf{j}))$, and $Q(\mathbf{j}) \in \mathrm{Mor}_{CK\mathbf{GA}}(W(\mathbf{j}), Y(\mathbf{j}))$.
We define their product $Q(\mathbf{j})P(\mathbf{j}) \in \mathrm{Mor}_{CK\mathbf{GA}}(V(\mathbf{j}), Y(\mathbf{j}))$ by the following rule:
(a) the product of null with any morphism is null: $\mathrm{null}_{W(\mathbf{j}),Y(\mathbf{j})} \cdot P(\mathbf{j}) = \mathrm{null}_{V(\mathbf{j}),Y(\mathbf{j})}$; $Q(\mathbf{j}) \cdot \mathrm{null}_{V(\mathbf{j}),W(\mathbf{j})} = \mathrm{null}_{V(\mathbf{j}),Y(\mathbf{j})}$; (b) if $P(\mathbf{j}) \neq \mathrm{null}$ and $Q(\mathbf{j}) \neq \mathrm{null}$ and

$$\ker Q(\mathbf{j}) \bigcap \mathrm{Indef} P(\mathbf{j}) = 0\,, \tag{9}$$

$$\mathrm{im} P(\mathbf{j}) + D(Q(\mathbf{j})) = W(\mathbf{j})\,, \tag{10}$$

then $Q(\mathbf{j})$ and $P(\mathbf{j})$ are multiplied as linear relations. Otherwise, $Q(\mathbf{j})P(\mathbf{j}) = \mathrm{null}$.
The category $CK\mathbf{GD}$. The *objects* of the category $CK\mathbf{GD}$ are the even–dimensional complex linear Cayley–Klein spaces $V(\mathbf{j})$ endowed with a non-degenerate symmetric bilinear form $\{.,.\}_{V(\mathbf{j})}$.
Let $V(\mathbf{j}), W(\mathbf{j}) \in \mathrm{Ob}(CK\mathbf{GD})$. We introduce in $V(\mathbf{j}) \oplus W(\mathbf{j})$ the bilinear form

$$\{(v,\omega),(v',\omega')\}_{V(\mathbf{j})\oplus W(\mathbf{j})} = \{v,v'\}_{V(\mathbf{j})} - \{\omega,\omega'\}_{W(\mathbf{j})}\,. \tag{11}$$

The morphisms from $V(\mathbf{j})$ to $W(\mathbf{j})$ in the category $CK\mathbf{GD}$ are of two types:
(a) maximal isotropic subspaces of $V(\mathbf{j}) \oplus W(\mathbf{j})$; (b) the formal element $\mathrm{null}_{V(\mathbf{j}),W(\mathbf{j})}$ (this element does not correspond to any subspace of $V(\mathbf{j}) \oplus W(\mathbf{j})$).
We need to define the product $Q(\mathbf{j})P(\mathbf{j})$ of morphisms $P(\mathbf{j}) \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$; and $Q(\mathbf{j}) \in \mathrm{Mor}(W(\mathbf{j}), Y(\mathbf{j}))$. First, the product of null and any other morphism is equal to null, that is,

$$\mathrm{null}_{W(\mathbf{j}),Y(\mathbf{j})} \cdot P = \mathrm{null}_{V(\mathbf{j}),Y(\mathbf{j})}\,; \quad Q(\mathbf{j}) \cdot \mathrm{null}_{V(\mathbf{j}),W(\mathbf{j})} = \mathrm{null}_{V(\mathbf{j}),Y(\mathbf{j})}\,. \tag{12}$$

for any $P(\mathbf{j}) \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$; and $Q(\mathbf{j}) \in \mathrm{Mor}(W(\mathbf{j}), Y(\mathbf{j}))$. Now let $P(\mathbf{j})$ and $Q(\mathbf{j})$ be linear relations. If $\ker(Q(\mathbf{j})) \cap \mathrm{Indef}P(\mathbf{j}) = 0$, then $Q(\mathbf{j})$ and $P(\mathbf{j})$ are multiplied as linear relations. If, on the other hand, (12) does not hold, then $Q(\mathbf{j})P(\mathbf{j}) = \mathrm{null}_{V(\mathbf{j}),Y(\mathbf{j})}$.
The Cayley–Klein category $CK\mathbf{A}$ is the category of linear Cayley–Klein spaces and linear operators.
The Cayley–Klein category $CK\mathbf{B}$ has odd-dimensional complex Cayley–Klein spaces as objects endowed with a non-degenerate symmetric bilinear form $\{.,.\}_{V(\mathbf{j})}$. Let $V(\mathbf{j}), W(\mathbf{j})$ be objects of the category $CK\mathbf{B}$. We introduce in $V(\mathbf{j}) \oplus W(\mathbf{j})$ the symmetric bilinear form

$$\{(v,\omega),(v',\omega')\}_{V(\mathbf{j})\oplus W(\mathbf{j})} = \{v,v'\}_{V(\mathbf{j})} - \{\omega,\omega'\}_{W(\mathbf{j})}\,. \tag{13}$$

The set $\mathrm{Mor}_{CK\mathbf{B}}(V(\mathbf{j}), W(\mathbf{j}))$ consists of null and maximal isotropic subspaces of $V(\mathbf{j}) \oplus W(\mathbf{j})$. The morphisms are multiplied in the same way as in the category $CK\mathbf{GA}$.

An object of the Cayley–Klein category $CK\mathbf{C}$ is a finite–dimensional complex linear Cayley–Klein space $V(\mathbf{j})$ endowed with a non-degenerate skew-symmetric bilinear form $\{.,.\}_{V(\mathbf{j})}$. If $V(\mathbf{j}), W(\mathbf{j}) \in \mathrm{Ob}_{CK\mathbf{C}}$, then we introduce in $V(\mathbf{j}) \oplus W(\mathbf{j})$ the skew-symmetric bilinear form defined by (13). The set $\mathrm{Mor}_{CK\mathbf{C}}(V(\mathbf{j}), W(\mathbf{j}))$ consists of null and all maximal isotropic subspaces of $V(\mathbf{j}) \oplus W(\mathbf{j})$. The morphisms are multiplied in the same way as in the Cayley–Klein category $CK\mathbf{GA}$.
The Cayley–Klein category $CK\mathbf{D}$. Let $Y(\mathbf{j})$ be a complex even-dimensional Cayley–Klein space endowed with a non-degenerate symmetric bilinear form. We denote by $\mathrm{Gr}(Y(\mathbf{j}))$ the Grassmannian of all maximal isotropic subspaces of $Y(\mathbf{j})$. An object of the category $CK\mathbf{D}$ is an object $V(\mathbf{j})$ of the category $CK\mathbf{GD}$ in which one of the connected components $\mathrm{Gr}_+(V(\mathbf{j})) = \mathrm{Mor}_{CK\mathbf{GD}}(0, V(\mathbf{j}))\backslash\mathrm{null}$ is fixed. Let $V(\mathbf{j}),\ W(\mathbf{j})$ be objects in $CK\mathbf{D}$. Let $W_+(\mathbf{j}) \in Gr_+(W(\mathbf{j})),\ V_+(\mathbf{j}) \in \mathrm{Gr}_+(V(\mathbf{j}))$, and let $W_-(\mathbf{j}) \in \mathrm{Gr}(W(\mathbf{j}))$ be a complement of $W_+(\mathbf{j})$ with respect to $W(\mathbf{j})$. The set $\mathrm{Mor}_{CK\mathbf{D}}(V(\mathbf{j}), W(\mathbf{j}))$ consists of null and the connected component of $\mathrm{Mor}_{CK\mathbf{GD}}(V(\mathbf{j}), W(\mathbf{j}))\backslash\mathrm{null} = \mathrm{Gr}(V(\mathbf{j}) \oplus W(\mathbf{j}))$ containing $V_+(\mathbf{j}) \oplus W_-(\mathbf{j})$. The morphisms are multiplied in the same way as in the category $CK\mathbf{GD}$.

LEMMA 1 *(a) Let* $P \in \mathrm{Mor}_{CK\mathbf{CK}}(V(\mathbf{j}), W(\mathbf{j}))$ *and* $Q \in \mathrm{Mor}_{CK\mathbf{CK}}(W(\mathbf{j}), Y(\mathbf{j}))$*, and suppose that* $QP \neq \mathrm{null}$ *. Then* $\dim Q(\mathbf{j})P(\mathbf{j}) = \dim P(\mathbf{j}) + \dim Q(\mathbf{j}) - \dim W(\mathbf{j})$.
(b) Multiplication of morphisms is a jointly continuous operation.

Proof (a) Let $H = V(\mathbf{j}) \oplus W(\mathbf{j}) \oplus W(\mathbf{j}) \oplus Y(\mathbf{j})$, and let Z be the subspace of vectors of the form (v, w, w, y). We define the subspace $T = P \oplus Q$ as the set of all vectors of the form (v, w, w', y), where $(v, w) \in Q$ and $(w', y) \in P$. In view of (10) we have $T + Z = H$. Thus $T \cap Z$ has dimension $\dim\, Z + \dim\, T - \dim\, H = \dim\, P + \dim\, Q - \dim\, W$.

Next we denote the projection of H onto $V(\mathbf{j}) \oplus Y(\mathbf{j})$ along $W(\mathbf{j}) \oplus W(\mathbf{j})$ by π. Then, as is easily seen, $\pi(T \cap Z)$ is the product QP. Furthermore, π is injective on $T \cap Z$. In fact, $\pi(v, w, w', y) = 0$ implies that $v = 0$ and $y = 0$, while $(v, w, w', y) \in Z$ implies that $w = w'$. Finally, $(0, w, w, 0) \in T$ implies that w lies in ker $\cap$ Indef P, which, by virtue of (9), consists merely of the origin. This completes the proof of assertion (a).
Assertion (b) is proved in similar fashion. □

LEMMA 2 *Let $P(\mathbf{j}) : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ and $Q(\mathbf{j}) : W(\mathbf{j}) \rightrightarrows Y(\mathbf{j})$ be morphisms of the Cayley–Klein category $CK\mathbf{K}$. Then*
(a) $(P'(\mathbf{j}))' = P(\mathbf{j})$; *(b)* $\dim P'(\mathbf{j}) + \dim P(\mathbf{j}) = \dim V(\mathbf{j}) + \dim W(\mathbf{j})$;
(c) $\ker P'(\mathbf{j}) = \mathrm{Ann} D(P(\mathbf{j}))$, $D(P'(\mathbf{j})) = \mathrm{Ann} \ker P(\mathbf{j})$, $\mathrm{Indef} P'(\mathbf{j}) = \mathrm{Ann}\ \mathrm{im} P(\mathbf{j})$, *and* $\mathrm{im} P'(\mathbf{j}) = \mathrm{Ann}\ \mathrm{Indef} P(\mathbf{j})$; *(d)* $Q'(\mathbf{j})P'(\mathbf{j}) = \mathrm{null}$ *in the category $CK\mathbf{CK}$ if and only if* $Q(\mathbf{j})P(\mathbf{j}) = \mathrm{null}$ *in the category $CK\mathbf{K}$;* *(e)* $(Q(\mathbf{j})P(\mathbf{j}))' = Q'(\mathbf{j})P'(\mathbf{j})$.

Proof (a) $\mathrm{Ann}\,\mathrm{Ann}\, T = T$. (b) and (c) are simple exercises in linear algebra. (d) Conditions (9) and (10) change places on passing to the dual morphism. (e) Let Q, P, and QP be non-null morphisms. Let $(f'', f) \in Q'P'$. Then there exists $f' \in W'(\mathbf{j})$ such that $(f'', f') \in Q'$ and $(f', f) \in P'$. Let (y, v) be an arbitrary element of QP. Then there exists $w \in W$ such that $(y, w) \in Q$ and $(v, w) \in P$. Then $f''(y) = f'(w) = f(v)$ by definition of the dual morphism. Thus (f'', f') belongs to $(QP)'$, and hence $Q'P' \subset (QP)'$. On the other hand, the dimensions of $Q'P'$ and $(QP)'$ are the same (see Lemma 1). This completes the proof. □

LEMMA 3 *Multiplications of morphisms of the $CK\mathbf{K}$ is associative.*

Proof Let $P \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$, $Q \in \mathrm{Mor}(W(\mathbf{j}), Y(\mathbf{j}))$, $R \in \mathrm{Mor}(Y(\mathbf{j}), Z(\mathbf{j}))$, and suppose that $R(QP) = \mathrm{null}$. This means that one of the following four conditions holds:
(1) $\ker\ Q \cap \mathrm{Indef}\ P \neq O$, (2) $\ker\ R \cap \mathrm{Indef}\ (QP) = 0$, (3) $\mathrm{im}\, P + D(Q) \neq W$,
(4) $D(R) + \mathrm{im}\, QP \neq Y$. Suppose first that the case (1) holds. Clearly $\ker(RQ) \supset \ker Q$, and therefore we have $\mathrm{Indef}\ P \cap \ker(RQ) \neq 0$. Hence $(RQ)R = \mathrm{null}$. Suppose next that the case (2) holds. Then $y \in \mathrm{Indef}(QP) \cap \ker R$ contains a non-zero vector. Take $w \in W$ such that $(0, w) \in P$ and $(w, y) \in Q$. Then $(y, 0) \in R$, which implies that $w \in \mathrm{Indef} P \cap \ker(RQ)$. Hence $(RQ)P = \mathrm{null}$. The cases (3) and (4) reduce to (1) and (2) by going over to the dual morphisms. □

PROPOSITION 1 *The definitions of the Cayley–Klein categories mentioned above are correct.*

Proof We have just verified associativity. Let be $QP \neq \mathrm{null}$, where $P \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$, $Q \in \mathrm{Mor}(W(\mathbf{j}), Y(\mathbf{j}))$. Then, by Lemma 1 $\dim\ QP = \frac{1}{2}(\dim\ V(\mathbf{j}) + \dim\ W(\mathbf{j}))$.
Hence the isotropic subspace QP has maximum possible dimension. □

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