Exact plasma equilibria from symmetries and transformations of MHD and CGL equilibrium equations

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Exact isotropic and anisotropic plasma equilibria are constructed as solutions to nonlinear 3D Magnetohydrodynamic (MHD) and anisotropic Chew–Goldberger–Low (CGL) plasma equilibrium equations, using the representation of equilibrium equations in coordinates connected with magnetic surfaces.

Infinite–dimensional symmetries of MHD and CGL equilibrium equations used in this construction are discussed from the prospective of Lie group analysis.

The infinite–parameter set of transformations between MHD and CGL equilibrium systems is employed to produce families of anisotropic (CGL) equilibria from particular isotropic (MHD) ones.

Solutions produced with the presented method are generally fully 3D solutions with no geometrical symmetries; they have different topologies and physical properties, and can serve as models of astrophysical phenomena.

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1 Introduction

I. Continuum plasma models. Description of plasma as a continuous medium is widely used in applications to controlled thermonuclear fusion, astrophysics (star formation, solar activity, earth magnetosphere, etc.), and terrestrial applications (laboratory and industrial plasmas.) Appropriate references are [1] – [14].

Two most popular continuum plasma models are the isotropic Magnetohydrodynamics (MHD) equations [15] and the anisotropic Chew–Goldberger–Low (CGL) equations [16]. These systems were obtained from Boltzmann and Maxwell equations under different assumptions [16, 17, 18].

In particular, the isotropic MHD system is valid when the mean free path of plasma particles is much less than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions. The system has the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0\,, \tag{1}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} + \mu_1 \triangle \mathbf{V}\,, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \triangle \mathbf{B}\,, \qquad \eta = \frac{1}{\sigma \mu}\,, \tag{3}$$

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$$\operatorname{div} \mathbf{B} = 0\,, \quad \mathbf{J} = \frac{1}{\mu} \operatorname{curl} \mathbf{B}\,. \tag{4}$$

Here $\mathbf{V}$ is plasma velocity, $\mathbf{B}$ is magnetic field, $\mathbf{J}$, electric current density, ρ, plasma density, μ, magnetic permeability of free space, σ, conductivity coefficient, μ_1, plasma viscosity coefficient; η, resistivity coefficient. The usual scalar Laplace operator is denoted by $\triangle$. The MHD system must be closed with an additional equation of state.

For a vanishing magnetic field, $\mathbf{B}$=0, the above system is reduced to Navier–Stokes equations of motion of a viscous fluid.

On the other hand, when the mean free path for particle collisions is long compared to Larmor radius (for instance, in strongly magnetized or rarified plasmas), the CGL model should be used. In this approach, the density function in Boltzmann equation is expanded in the powers of the Larmor radius [16].

The resulting system is anisotropic, because it has a distinguished direction — the direction of the magnetic field $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0\,, \tag{5}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{div} \mathcal{P} - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} + \mu_1 \triangle \mathbf{V}\,, \tag{6}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \triangle \mathbf{B}\,, \qquad \eta = \frac{1}{\sigma \mu}\,, \tag{7}$$

$$\operatorname{div} \mathbf{B} = 0\,, \qquad \mathbf{J} = \frac{1}{\mu} \operatorname{curl} \mathbf{B}\,, \tag{8}$$

$$\mathcal{P}_{ij} = p_\perp \delta_{ij} + \tau B_i B_j\,, \quad \tau = \frac{p_\parallel - p_\perp}{\mathbf{B^2}}\,, \quad i, j = 1, 2, 3\,. \tag{9}$$

$\mathcal{P}$ is a 3×3 pressure tensor with two independent components: the pressure along the magnetic field $p_\parallel$ and in the transverse direction $p_\perp$. τ is the anisotropy factor.

For the CGL system to be closed, one needs to add to it two equations of state.

The Boltzmann equation used with other assumptions lets one develop different plasma continuum models, for example, multi–fluid ones, where different kinds of particles are looked at as independent fluids [18]. However the MHD and CGL models have received most attention.

From now on we restrict ourselves to the case of nonviscous infinitely–conducting plasmas: $\mu_1 = \eta = 0$. This approximation is natural in the case of large kinetic and magnetic Reynolds numbers. Under this assumption, both MHD and CGL systems have several remarkable analytical properties. In particular, one can name, for both systems, the "frozen-in magnetic field" property (Kelvin's theorem), Lagrangian and Hamiltonian formulation [19], and conservation of helicity [20].

II. Ideal plasma equilibrium models. In many applications, dynamic and stationary *equilibrium* states of plasmas are of particular importance.

This work continues a series of publications [6], [21], [22], [23] – [29], devoted to the study of symmetries and exact solutions of equilibrium states of the isotropic (MHD) the anisotropic (CGL) magnetohydrodynamics equations.

Ideal MHD equilibrium equations are obtained as a time–independent reduction of the MHD system (1) – (4) in the case of infinite conductivity and negligible viscosity:

$$\rho\mathbf{V}\times\operatorname{curl}\mathbf{V}-\frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B}-\operatorname{grad}P-\rho\operatorname{grad}\frac{\mathbf{V}^2}{2}=0\,, \tag{10}$$

$$\operatorname{div}\big(\rho\mathbf{V}\big)=0\,,\quad \operatorname{curl}(\mathbf{V}\times\mathbf{B})=0\,,\quad \operatorname{div}\mathbf{B}=0\,. \tag{11}$$

In this paper we discuss incompressible plasmas:

$$\operatorname{div}\mathbf{V}=0. \tag{12}$$

Incompressibility condition is a good approximation for subsonic plasma flows with low Mach numbers $M\ll 1$, $M^2=\mathbf{V}^2/(\gamma P/\rho)$. For incompressible plasmas, the continuity equation $\operatorname{div}\rho\mathbf{V}=0$ implies $\mathbf{V}\cdot\operatorname{grad}\rho=0$, hence density is constant on streamlines.

In the particular case of *static plasma equilibrium* ($\mathbf{V}=0$), the system takes the form

$$\operatorname{curl}\mathbf{B}\times\mathbf{B}=\mu\operatorname{grad}P\,,\quad \operatorname{div}\mathbf{B}=0\,. \tag{13}$$

An important reduction of this system is the constant–pressure *force-free plasma equilibrium system*

$$\operatorname{curl}\mathbf{B}=\alpha(\mathbf{r})\mathbf{B}\,,\quad \operatorname{div}\mathbf{B}=0\,. \tag{14}$$

All static and dynamic incompressible MHD equilibria (except Beltrami flows $\operatorname{curl}\mathbf{B}=\alpha\mathbf{B}$, $\alpha=\text{const}$) are known to possess a family of *magnetic surfaces* (or a foliation) $\Psi(\mathbf{r})=\text{const}$, to which velocity both $\mathbf{V}$ and $\mathbf{B}$ are tangent, and thus magnetic field lines and plasma streamlines lie on these surfaces [30, 21, 22].

The equilibrium reduction of *anisotropic* (CGL) equations (5) – (9) is [26, 27]

$$\begin{aligned}\rho\mathbf{V}\times\operatorname{curl}\mathbf{V}-\left(\frac{1}{\mu}-\tau\right)\mathbf{B}\times\operatorname{curl}\mathbf{B}=\operatorname{grad}\ p_\perp+\rho\operatorname{grad}\frac{\mathbf{V}^2}{2}+\\ +\tau\operatorname{grad}\frac{\mathbf{B}^2}{2}+\mathbf{B}(\mathbf{B}\cdot\operatorname{grad}\tau)\,,\end{aligned} \tag{15}$$

$$\operatorname{div}\big(\rho\mathbf{V}\big)=0\,,\quad \operatorname{curl}(\mathbf{V}\times\mathbf{B})=0\,,\quad \operatorname{div}\mathbf{B}=0\,. \tag{16}$$

For this system to be closed, one needs to add to it two equations of state. In this paper we will consider incompressible CGL plasmas: $\operatorname{div}\mathbf{V}=0$.

In [27], we have shown that CGL equilibrium equations generally share the topology of the MHD equilibrium system (10), (11): the plasma domain is also spanned by nested 2D magnetic surfaces. (This may not be the case only when $\mathbf{B}||\mathbf{V}$.)

The *static* CGL equilibrium equations have the form is (cf. (13))

$$\left(\frac{1}{\mu} - \tau\right) \operatorname{curl} \mathbf{B} \times \mathbf{B} = \operatorname{grad} p_{\perp} + \tau \operatorname{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \operatorname{grad} \tau)\,, \quad \operatorname{div} \mathbf{B} = \mathbf{0}\,. \quad (17)$$

Both the isotropic MHD equilibrium system (10) – (12) and the anisotropic CGL equilibrium system (15), (16), together with their static reductions (13) and (17), are essentially non-linear systems of partial differential equations depending on three spatial variables. No general methods exist for the construction of exact solutions to the corresponding boundary value problems in generic domains; the question of stability is answered only for particular types of instabilities (for a review, see [27].) However, exact solutions, especially those having natural physical behaviour (see [26]), are demanded by applications.

In the present paper, we construct new MHD and CGL equilibria using the recently discovered analytical properties of the above systems — infinite symmetries and transformations of incompressible MHD equilibria into solutions to incompressible CGL equilibrium equations.

In Section 2, we discuss analytical properties of ideal incompressible MHD and CGL equilibrium systems, in particular, infinite–parameter Lie groups of symmetries and their connection with classical Lie group analysis of these systems. It is remarkable that these groups, the richest known classes of transformations for these systems, are equivalent to certain Lie point transformations of corresponding systems. Infinite *transformations* between solutions of MHD and CGL equilibrium systems provide a tool for the construction of sets of anisotropic equilibrium configurations from isotropic ones, the former retaining the topology and boundedness of the latter and having certain stability properties.

In Section 3, we present a method of building exact 3D isotropic and anisotropic plasma equilibria with and without dynamics, and often without geometrical symmetry, in different geometries and with physically relevant properties. The method is based on representing the system of static classical plasma equilibrium (13) equations in coordinates connected with magnetic surfaces. In many important cases, for a given set of magnetic surfaces, an *orthogonal coordinate system* can be chosen, with one of the coordinates constant on the magnetic surfaces. In such coordinates, the static plasma equilibrium system is reduced to two partial differential equations for two unknown functions. The suggested representation is used for producing particular exact static solutions in different geometries. These solutions, by virtue of symmetries and transformations discussed in Section 2, give rise to *families* of more complicated dynamic and static, isotropic and anisotropic equilibrium configurations.

A particular example that illustrates the use of the introduced approach is given. We construct exact dynamic isotropic and anisotropic plasma equilibrium configurations that model *solar flares* in the star coronal plasma near an active region. The resulting model is essentially non-symmetric and presented in an exact form; it reproduces important features of solar flares known from observations. The presented equilibrium has ellipsoidal magnetic surfaces.

Several other examples of exact solutions constructed using the coordinate representation method are found in [27].

2 Symmetries and transformations of plasma equilibrium systems

2.1 Symmetry properties of plasma equilibrium systems

In this section, we list known symmetries and transformations of MHD and CGL equilibrium equations. ("Symmetry" here means that a system of differential equations is invariant under a certain change of variables.)

Historically, *reflection* and *interchange* symmetries were observed first.

i. Reflection symmetry. The general system of equations of compressible and incompressible MHD and CGL equilibria (10), (11), (15), (16) admit the following two independent reflection symmetries:

$$\mathbf{V} \to -\mathbf{V}\,, \quad \mathbf{B} \to -\mathbf{B}\,.$$

ii. "Interchange symmetry". If the density $\rho = \text{const}$, then by a scaling transform $\mathbf{V}_1 = \sqrt{\rho}\,\mathbf{V}$, $\mathbf{B}_1 = \sqrt{1/\mu}\,\mathbf{B}$ the MHD equilibrium system (10), (11) can be rewritten in the invariant form [31]:

$$\begin{aligned}
&\mathbf{V}_1 \times \operatorname{curl}\mathbf{V}_1 - \mathbf{B}_1 \times \operatorname{curl}\mathbf{B}_1 - \operatorname{grad} P_1 = 0\,,\\
&\operatorname{curl}(\mathbf{V_1} \times \mathbf{B}_1 = 0\,, \quad \operatorname{div}\mathbf{B}_1 = 0\,, \quad \operatorname{div}\mathbf{V}_1 = 0\,.
\end{aligned}$$

If a solution to this system $\{\mathbf{V}_1, \mathbf{B}_1, P_1\}$ is known, then evidently $\{\mathbf{B}_1, \mathbf{V}_1, -P_1\}$ is also a solution, i.e. the system is invariant under the transformation

$$\mathbf{V} \longleftrightarrow \mathbf{B}\,, \quad P \to -P\,.$$

The CGL equilibrium system possesses the same symmetries when $\tau = \text{const}$, $\tau < 1/\mu$ [27].

iii. Infinite symmetries of MHD equilibrium equations. It was recently shown by Bogoyavlenskij [21, 22] that ideal isotropic MHD equilibrium equations (10)−(12) possess a family of intrinsic symmetries. If $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ is a solution of (10), (11), where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines and streamlines, then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), P_1(\mathbf{r}), \rho_1(\mathbf{r})\}$ is also a solution, where

$$\begin{aligned}
\mathbf{B}_1 &= b(\mathbf{r})\mathbf{B} + c(\mathbf{r})\sqrt{\mu\rho}\,\mathbf{V}\,,\\
\mathbf{V}_1 &= \frac{c(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho}}\,\mathbf{B} + \frac{b(\mathbf{r})}{a(\mathbf{r})}\,\mathbf{V}\,,\\
\rho_1 = a^2(\mathbf{r})\rho\,, \quad P_1 &= CP + \frac{C\mathbf{B}^2 - \mathbf{B}_1^2}{2\mu}\,.
\end{aligned} \tag{18}$$

Here $b^2(\mathbf{r}) - c^2(\mathbf{r}) = C = \text{const}$, and $a(\mathbf{r})$, $b(\mathbf{r})$, $c(\mathbf{r})$ are *arbitrary* functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi(\mathbf{r}) = \text{const}$, when they exist).

These symmetries form an infinite–dimensional Abelian group [22]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2\,. \tag{19}$$

Bogoyavlenskij symmetries do not change the topology (the set of magnetic surfaces for non-field–aligned solutions, and field lines for field–aligned solutions), and preserve the Lagrangian [21, 22]. The symmetries depend totally on two arbitrary functions defined on a cellular complex depending on the initial solution topology (see [21, 22, 27].)

iv. Infinite symmetries of CGL equilibrium equations. In [29, 25, 26, 27] we have proven that the system of incompressible anisotropic (CGL) equilibrium equations (15), (16), (12) possesses a similar family of intrinsic symmetries: for each solution $\{\mathbf{V}(\mathbf{r}),\ \mathbf{B}(\mathbf{r}),\ p_\perp(\mathbf{r}),\ p_\|(\mathbf{r}),\ \rho(\mathbf{r})\}$ with density $\rho(\mathbf{r})$ and anisotropy factor $\tau(\mathbf{r})$ (9) constant on both magnetic field lines and streamlines, the functions

$$\begin{gathered}
\rho_1 = m^2(\mathbf{r})\rho\,,\\
\mathbf{V}_1 = \frac{b(\mathbf{r})\sqrt{1/\mu-\tau}}{m(\mathbf{r})\sqrt{\rho}}\,\mathbf{B} + \frac{a(\mathbf{r})}{m(\mathbf{r})}\,\mathbf{V}\,,\\
\mathbf{B}_1 = \frac{a(\mathbf{r})}{n(\mathbf{r})}\,\mathbf{B} + \frac{b(\mathbf{r})\sqrt{\rho}}{n(\mathbf{r})\sqrt{1/\mu-\tau}}\,\mathbf{V}\,,\\
p_{\perp 1} = Cp_\perp + \frac{C\mathbf{B}^2-\mathbf{B_1^2}}{2\mu}\,,\\
p_{\|1} = p_\| n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2} + p_\perp\left(C - n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2}\right) + \frac{C\mathbf{B}^2+\mathbf{B}_1^2(1-2n^2(\mathbf{r}))}{2\mu}\,.
\end{gathered} \tag{20}$$

also define an infinite family of solutions, where $a^2(\mathbf{r}) - b^2(\mathbf{r}) = C = \text{const}$, and $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ are functions constant on both magnetic field lines and streamlines.

These symmetries also form a Lie group [26]

$$G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \tag{21}$$

that contains the group (19) of Bogoyavlenskij symmetries of MHD equations.

Like Bogoyavlenskij symmetries for the isotropic case, the symmetries (20) do not affect the solution topology, preserve the corresponding Lagrangian, and do not introduce fire–hose instability [26, 27]. They depend on three arbitrary functions, whose domain is determined by the initial solution topology [26, 27].

2.2 Connection of infinite symmetries with Lie group analysis

It has long been known that the Lie group analysis method (e.g. [32]) is applicable to any (sufficiently smooth) ODE/PDE system, and is generally capable of detecting both its simple geometric symmetries (e.g., rotations, scaling transforms and translations), and more complicated ones. However, this method is extremely resource–demanding, therefore in most applications, symbolic computation software is used (for a review, see [23, 33].)

The direct Lie group analysis of full MHD and CGL equilibrium systems, without additional assumptions, has not been undertaken, due to high numbers of variables and resulting complexity even for modern symbolic computation software.

However, the analysis of the *static plasma equilibrium system* (13) has revealed that it is invariant only under the spatial shifts/rotations and space/parameter uniform scaling [28]; these transformations cannot be used to produce plasma equilibrium configurations of new types.

The question about the possibility of obtaining the infinite-parameter Lie group of Bogoyavlenskij symmetries (18), using Lie formalism, was raised soon after the discovery of the symmetries. (For instance, Lie symmetries listed in the book [34] for the general viscous time–dependent MHD system do not contain relevant infinitesimal generators.)

In [23], we have shown that the Bogoyavlenskij symmetries can indeed be found as Lie point transformations of the MHD equilibrium system *only* if general solution topology (the existence of magnetic surfaces to which vector fields $\mathbf{B}$ and $\mathbf{V}$ are tangent) and the incompressibility condition are explicitly taken into account in the form of additional constraints:

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r}))\,, \quad \operatorname{grad}(\Psi(\mathbf{r}))\cdot\mathbf{B} = 0\,, \quad \operatorname{grad}(\Psi(\mathbf{r}))\cdot\mathbf{V} = 0\,.$$

Here $\Psi(\mathbf{r})$ is a magnetic surface function (or, more generally, a function constant on magnetic field lines and plasma streamlines.)

The infinitesimal generators giving rise to the Bogoyavlenskij transformations (18) are [23]

$$\begin{aligned}
X^{(1)} &= M(\mathbf{r})\left(\sum_{k=1}^{3}\frac{B_k}{\mu\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k\frac{\partial}{\partial B_k} - \frac{1}{\mu}(\mathbf{V}\cdot\mathbf{B})\frac{\partial}{\partial P}\right),\\
X^{(2)} &= \sum_{k=1}^{3} V_k\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k\frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial P}\,,\\
X^{(3)} &= N(\mathbf{r})\left(2\rho\frac{\partial}{\partial\rho} - \sum_{k=1}^{3} V_k\frac{\partial}{\partial V_k}\right).
\end{aligned}$$

The infinite–dimensional Lie group of symmetries (20), (21) of incompressible *anisotropic* (CGL) plasma equilibrium equations (15), (16), (12) can also be found from Lie group analysis, as shown in [27]. This can only be done if, as in the above MHD case, a function $\Psi(\mathbf{r})$ constant on magnetic field lines and plasma streamlines is introduced, and the system is extended with constraints $\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r}))$, $\tau(\mathbf{r}) = \tau(\Psi(\mathbf{r}))$. Then the symmetries (20) are equivalent to Lie point transformations produced by infinitesimal generators [27]

$$X^{(4)} = M(\mathbf{r})\left(\sum_{k=1}^{3} B_k\frac{1/\mu-\tau}{\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k\frac{\partial}{\partial B_k} - \frac{1}{\mu}(\mathbf{V}\cdot\mathbf{B})\frac{\partial}{\partial p_\perp}\right),$$

$$
\begin{aligned}
X^{(5)} &= \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2p_\perp \frac{\partial}{\partial p_\perp}\,, \\
X^{(6)} &= N(\mathbf{r}) \left(2\rho \frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} \right), \\
X^{(7)} &= L(\mathbf{r}) \left(2\left(\frac{1}{\mu} - \tau\right) \frac{\partial}{\partial \tau} - \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + \frac{B^2}{\mu} \frac{\partial}{\partial p_\perp} \right).
\end{aligned}
$$

Here and above $L(\mathbf{r})$, $M(\mathbf{r})$, $N(\mathbf{r})$ are arbitrary smooth functions constant on both magnetic field lines and streamlines.

Thus it was shown that such complex and important sets of transformations infinite–dimensional groups of Bogoyavlenskij symmetries (18), (20) of the isotropic (MHD) and anisotropic (CGL) plasma equilibrium equations, the richest known classes of transformations for these systems, can be obtained through the application of the general Lie algorithm.

2.3 The infinite family of transformations from MHD to CGL equilibria

In [25, 26, 27] we have found and discussed an infinite–dimensional set of transformations that connect solutions to incompressible MHD and CGL plasma equilibrium systems. The following theorem holds:

Theorem 1 *Let $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ be a solution of the system (10) – (12) of incompressible MHD equilibrium equations, where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines and plasma streamlines (i.e. on magnetic surfaces $\Psi = \mathrm{const}$, if they exist.)*

Then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\parallel 1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ is a solution to incompressible CGL plasma equilibria (15), (16), (12), where

$$
\begin{aligned}
\mathbf{B}_1(\mathbf{r}) = f(\mathbf{r})\mathbf{B}(\mathbf{r})\,, \quad \mathbf{V}_1(\mathbf{r}) &= g(\mathbf{r})\mathbf{V}(\mathbf{r})\,, \quad \rho_1 = C_0 \frac{\rho(\mathbf{r})\mu}{g^2(\mathbf{r})}\,, \\
p_{\perp 1}(\mathbf{r}) &= C_0 \mu P(\mathbf{r}) + C_1 + \left(C_0 - \frac{f^2(\mathbf{r})}{\mu} \right) \frac{\mathbf{B^2}(\mathbf{r})}{2}\,, \\
p_{\parallel 1}(\mathbf{r}) &= C_0 \mu P(\mathbf{r}) + C_1 - \left(C_0 - \frac{f^2(\mathbf{r})}{\mu} \right) \frac{\mathbf{B}^2(\mathbf{r})}{2}\,,
\end{aligned}
\tag{22}
$$

and $f(\mathbf{r})$, $g(\mathbf{r})$ are arbitrary functions constant on the magnetic field lines and streamlines. C_0, C_1 are arbitrary constants.

Under the conditions of the theorem, the anisotropy factor

$$
\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{\mathbf{B_1}^2} = \frac{1}{\mu} - \frac{C_0}{f^2(\mathbf{r})}
$$

is also constant on the magnetic field lines and streamlines.

The transformations (22) do not change the topology of plasma equilibrium configurations. All CGL solutions obtained from non-Beltrami MHD equilibria using Theorem 1 have the same magnetic surfaces as the original solution. In [27], we have shown that anisotropic solutions constructed using the transformations (22) can be made free of the fire-hose instability (and, in the static case, of the mirror instability) by the proper choice of the transformation parameters.

3 Exact isotropic and anisotropic plasma equilibria arising from the representation connected with magnetic surfaces

Bogoyavlenskij symmetries (18) and MHD→CGL transformations (22) have been extensively used to construct new solutions from known ones [21, 22, 26, 27]. These methods are capable of changing magnitudes of physical parameters of plasma, producing dynamic configurations from static ones, "mixing" velocity and magnetic field (if they are not parallel in the initial solution), but *not* capable of changing solution topology or infiniteness of its total energy (when it is infinite).

Therefore a need is observed in exact particular plasma equilibrium solutions with various domains, topologies and relations between physical parameters. Combined with infinite–dimensional symmetries and transformations discussed above, they would serve to produce realistic models of physical phenomena that involve plasma equilibrium.

Only several different examples and classes of exact plasma equilibria have been constructed so far (for a review, see [27]); many of them were found by dimension reduction methods (e.g. from Grad–Shafranov or JFKO equations) and therefore have geometrical symmetries. We also note that many known exact solutions have limited applicability in modelling because of the violation of necessary physical conditions (see [26]).

In this work, we suggest a method of building exact particular solutions in different geometries and with different sets of magnetic surfaces.

We start from representing the system of *static classical plasma equilibrium* equations (13) in coordinates connected with magnetic surfaces. In many important cases, for a given set of magnetic surfaces, an *orthogonal coordinate system* can be constructed, with one of the coordinates constant on the magnetic surfaces.

A set of coordinates is defined by its metric tensor coefficients; we establish sufficient conditions for the metric coefficients under which exact solutions to plasma equilibrium equations can be found. We also prove that in coordinates where the Laplace equation admits 2–dimensional solutions, non-trivial exact plasma equilibria of a certain type can be built.

In many systems of coordinates, classical and non-classical, non-trivial *gradient* vector fields can be found, tangent to prescribed sets of magnetic surfaces (see [29, 25].) Though gradient fields by themselves represent only degenerate plasma equilibria with constant pressure and no electric currents, and cannot model physical phenomena, they can serve as initial solutions in infinite–parameter transformations (such as (18), (22)) that produce non-trivial dynamic and static, isotropic and anisotropic plasma equilibrium configurations.

Examples of new exact isotropic and anisotropic plasma equilibrium solutions obtained using the above-described machinery are readily constructed. A configuration with elliptic magnetic surfaces described in section 3.2 below models isotropic and anisotropic solar flares. Other exact solutions, including non-symmetric plasma equilibria with spherical or non-circular–cylindrical magnetic surfaces and an astrophysical model of mass exchange between two spheroidal objects, are found in the thesis [27].

3.1 Plasma equilibrium equations in coordinates connected with magnetic surfaces

The question whether it is possible, given a family of smooth surfaces $A(x, y, z) =$ const in $\boldsymbol{R}^3$, to construct (at least locally) two other families of surfaces so that the three families form a triply orthogonal system, was answered by Darboux [35]: *a family of surfaces* $A(x, y, z) =$ const *is a part of a triply orthogonal system if and only if the function* $A(\mathbf{r})$ *satisfies a certain nonlinear partial differential equation of order 3* [35, 36]. Such families of surfaces were called by Darboux *the families of Lamé*. Thus, for a given family of Lamé, one can construct a system of orthogonal coordinates with one of the coordinates constant on surfaces $A(x, y, z) =$ const. There exist many examples of families of Lamé; they include sets of parallel surfaces; sets of surfaces of revolution; Ribaucour surfaces, and other families [36].

In orthogonal coordinates, the metric tensor is

$$g_{ij} = {H_i}^2 \delta_{ij}\,, \quad {H_i}^2 = \left(\frac{\partial x}{\partial u^i}\right)^2 + \left(\frac{\partial y}{\partial u^i}\right)^2 + \left(\frac{\partial z}{\partial u^i}\right)^2, \quad i, j = 1, 2, 3\,, \tag{23}$$

where H_i are the scaling (Lamé) coefficients.

All subscripts used below mean corresponding partial derivatives.

The following theorem contains a representation of static plasma equilibrium equations (13) in coordinates connected with magnetic surfaces.

Theorem 2 *To every solution* $\{\Phi(u, v, w),\ P(w)\}$ *of the system*

$$\frac{\partial}{\partial u}\left(\frac{\sqrt{g_{22}}\sqrt{g_{33}}}{\sqrt{g_{11}}}\,\Phi_u\right) + \frac{\partial}{\partial v}\left(\frac{\sqrt{g_{11}}\sqrt{g_{33}}}{\sqrt{g_{22}}}\,\Phi_v\right) = 0\,, \tag{24}$$

$$\frac{1}{g_{11}}\,\Phi_u\Phi_{uw} + \frac{1}{g_{22}}\,\Phi_v\Phi_{vw} = -P_w \tag{25}$$

in some orthogonal coordinates (u, v, w) *with a metric tensor* g_{ij} *there corresponds a solution to the isotropic static Plasma Equilibrium system (13) with magnetic surfaces* $w =$ const *forming a family of Lamé, the pressure* $P = P(w)$*, and the magnetic field*

$$\mathbf{B} = \frac{\Phi_u}{\sqrt{g_{11}}}\,\mathbf{e_u} + \frac{\Phi_v}{\sqrt{g_{22}}}\,\mathbf{e_v}\,. \tag{26}$$

For the sake of shortness, we do not reproduce the proofs of this and some of the following statements; they are found in the thesis [27].

Remark 1. The converse is also true: given a static MHD equilibrium (13) with magnetic surfaces being surfaces of Lamé, it can be shown that it satisfies the system (24), (25) [27].

The expression (24) is the (u, v)–part of the Laplace's equation in the coordinates (u, v, w). Therefore the system of static MHD equilibrium equations (13) with magnetic surfaces being surfaces of Lamé is (at least locally) equivalent to the system

$$\triangle_{(u,v)}\Phi = 0\,, \tag{27}$$

$$\mathrm{grad}_{(u,v)}\,\Phi \cdot \mathrm{grad}_{(u,v)}\,\Phi_w = -P_w\,, \tag{28}$$

where the subscript (u, v) means that only $u-$ and $v-$ parts of the corresponding differential operators are used.

Remark 2. In coordinate systems where $g_{11} = g_{11}(u, v)$, $g_{22} = g_{22}(u, v)$, the second equation of the system, (28), has a simple energy–connected interpretation. Indeed, by (26) it becomes

$$\frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial w}\left(\frac{\mathbf{B}^2}{2} + P\right) = 0\,.$$

For incompressible plasma equilibria, it means that *the component of the gradient of total energy density in the direction normal to the magnetic surfaces vanishes.* Therefore for any MHD equilibrium configuration in which magnetic surfaces $w =$ const form a family of Lamé, and where $g_{11} = g_{11}(u, v)$, $g_{22} = g_{22}(u, v)$, *the total energy can be finite only if the plasma domain is bounded in the direction transverse to magnetic surfaces.*

Remark 3. The electric current density $\mathbf{J} = \mu^{-1}\,\mathrm{curl}\,\mathbf{B}$ is written in terms of Φ as follows:

$$\mathbf{J} = \frac{1}{\mu}\left(-\frac{1}{\sqrt{g_{22}}\sqrt{g_{33}}}\frac{\partial^2\Phi}{\partial v\partial w}\,\mathbf{e_u} + \frac{1}{\sqrt{g_{11}}\sqrt{g_{33}}}\frac{\partial^2\Phi}{\partial u\partial w}\,\mathbf{e_v}\right). \tag{29}$$

Remark 4. As noted by Lundquist [37], the static MHD equilibrium equations (13) are equivalent to the time–independent incompressible Euler equations that describe ideal fluid equilibria. Therefore static Euler equations may also be presented in the form (24), (25).

Remark 5. As illustrated below and in the thesis [27], in many cases appropriate orthogonal coordinates (u, v, w) required by the above theorem may be introduced *globally* in the plasma domain $\mathcal{D}$.

Remark 6. A theorem proven in [27] gives *sufficient conditions* on the metric tensor components, such that in the corresponding coordinate system (u, v, w) solutions of the plasma equilibrium system (24), (25) of certain forms exist.

The following theorem shows how to construct trivial "vacuum" (gradient) magnetic fields in any coordinate system which affords 2–dimensional solutions to the Laplace equation.

Theorem 3 *In any coordinate system where the 3D Laplace equation*

$$\triangle_{(u,v,w)}\phi(u,v,w)=0$$

admits a solution independent of one of the variables (w), there exists a trivial ("vacuum") magnetic field configuration

$$\operatorname{div}\mathbf{B}=0\,,\quad \operatorname{curl}\mathbf{B}=0 \tag{30}$$

corresponding to this solution, and this magnetic field is tangent to surfaces $w=$ const.

Proof. If a solution of the Laplace equation $\phi(u,v)$ independent of w is given,

$$\frac{\partial}{\partial u}\frac{H_vH_w}{H_u}\frac{\partial\phi(u,v)}{\partial u}+\frac{\partial}{\partial v}\frac{H_uH_w}{H_v}\frac{\partial\phi(u,v)}{\partial v}+\frac{\partial}{\partial w}\frac{H_uH_v}{H_w}\frac{\partial\phi(u,v)}{\partial w}=0,$$

then it is indeed at the same time a solution to the "truncated" Laplace equation (27).

It also is a solution to the second plasma equilibrium equation (28) when $P(w)=$ const, because it nulls the left-hand side identically. Thus $\phi(u,v)$ defines a force-free plasma magnetic field (26).

From the fact that $\phi(u,v)$ satisfies the system (27), (28) it follows that the magnetic field (26) is tangent to the coordinate surfaces $w=$ const, by derivation of the equations.

For every such solution the electric current $\mathbf{J}$ (29) vanishes, thus making such plasma equilibrium a *vacuum magnetic field* configuration.

The theorem is proven.

Remark 1. Use of "vacuum" magnetic fields. As plasma equilibria, vacuum magnetic fields are *trivial* and can not be used for direct modelling of real physical equilibrium phenomena, where electric current $\mathbf{J}$, plasma pressure P and velocity $\mathbf{V}$ are generally non-zero. However, they can be used as an initial solution to construct new *non-trivial* solutions to Plasma Equilibrium equations in static and dynamic cases, for isotropic and anisotropic plasmas.

Indeed, the application of Bogoyavlenskij symmetries (18) to such a configuration results in non-trivial *field–aligned isotropic MHD solutions* with $P\neq$ const, $\mathbf{J}\neq 0$, and density being an arbitrary function of magnetic surface variable.

By the application of the MHD→CGL transforms (22) to a static vacuum magnetic field configuration, static anisotropic CGL plasma equilibria are obtained, also non-trivial in the sense $p_\parallel$, $p_\perp\neq$ const, $\mathbf{J}\neq 0$ and having the same topology as the original vacuum field. These static anisotropic equilibria can be extended further on the dynamic case with the help of the analog of Bogoyavlenskij symmetries for CGL plasmas (20) (see examples in subsection 3.2 and [27].)

Remark 2. Availability of vacuum magnetic fields. By Theorem 3, the solutions to the "truncated" Laplace's equation (27) are available in any coordinate system that allows simple separability of Laplace's equation or where two–dimensional solutions exist. Many classical and esoteric coordinate systems *do* admit two–dimensional solutions of the Laplace equation, as found in literature, for example, [38].

New systems of coordinates may be constructed where the Laplace's equation will be separable or have two–dimensional solutions. The list of necessary and sufficient conditions on the metric coefficients is available in [39].

3.2 An example of exact plasma equilibria

In this section, we present a particular example that illustrates the use of the above-described approach connected with magnetic surface representation for building new dynamic and static, isotropic and anisotropic plasma equilibria.

We construct exact dynamic isotropic and anisotropic plasma equilibrium configurations that model *solar flares* in the coronal plasma near an active region. The resulting model is essentially non-symmetric and presented in an exact form; it reproduces important features of solar flares known from observations. The presented equilibrium has ellipsoidal magnetic surfaces.

We start from the construction of vacuum magnetic fields tangent to ellipsoids, using the magnetic–surface–connected representation of plasma equilibria equations (Theorems 2 and 3). Then transformations are applied to this trivial solution to produce non-trivial isotropic and anisotropic plasma equilibria.

Ellipsoidal coordinates (η, θ, λ) are described in, e.g., [38]. The coordinate surfaces are ellipsoids and one–sheet and two–sheet hyperboloids.

Laplace's equation is separable in ellipsoidal coordinates, and we take a solution depending only on (θ, λ), so that its gradient has zero η–projection transverse to ellipsoids, but is tangent to them:

$$\Phi_1(\theta, \lambda) = \left[A_1 + B_1 \mathrm{sn}^{-1}\left(\sqrt{\frac{c^2 - \theta^2}{c^2 - b^2}}, \sqrt{\frac{c^2 - b^2}{c^2}} \right) \right] \cdot \left[A_2 + B_2 \mathrm{sn}^{-1}\left(\frac{\lambda}{b}, \frac{b}{c} \right) \right].$$

Here $\mathrm{sn}(x, k)$ is the Jacobi elliptic sine function. The inverse of it is an incomplete elliptic integral

$$F_{\mathrm{ell}}(z, k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}}.$$

b, c are the parameters of elliptic coordinate systems used for this solution; the notation agrees with [38].

$\Phi_1(\theta, \lambda)$ does not depend on η, and therefore evidently satisfies both equations (24), (25) (with $u = \theta,\ v = \lambda$.) The resulting magnetic field (26) is tangent to ellipsoids $\eta = \mathrm{const}$, and has a singularity at $\theta = \lambda$, i.e. on the plane $y = 0$.

However one may verify that for a plasma region $c < \eta_1 < \eta < \eta_2$ the total magnetic energy $\int_V \left(B^2/2\right) \mathrm{d}v$ is finite. Also, if one restricts to a half-space $y > 0$

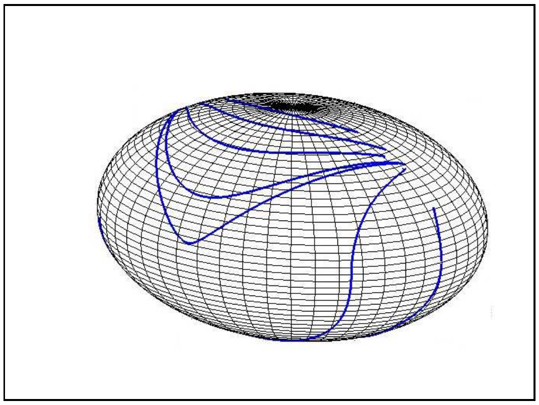

Fig. 1. Lines of the magnetic field (32) tangent to the ellipsoid $\eta = 12$. The shown ellipsoid is a magnetic surface from the family of nested ellipsoids $\eta = \mathrm{const}$ in classical ellipsoidal coordinates. This configuration is smoothly defined in the half–space $y > 0$.

or $y < 0$, then the magnetic field is well-defined in a continuous and differentiable way.

If the magnetic field is tangent to the boundary of a domain, one can safely assume that outside of it $\mathbf{B} = 0$ identically. This is achieved, as usual, by introducing a boundary surface current

$$\mathbf{i}_{\mathrm{b}}(\mathbf{r}_1) = \mu^{-1}\mathbf{B}(\mathbf{r}_1) \times \mathbf{n}_{\mathrm{out}}(\mathbf{r}_1)\,, \tag{31}$$

where $\mathbf{r}_1$ is a point on the boundary of the domain, and $\mathbf{n}_{\mathrm{out}}$ is an outward normal.

Fig. 1 shows several magnetic field lines for the case $(b = 7, c = 10, A_1 = A_2 = 0, B_1 = 1/100, B_2 = 1/30)$ on the ellipsoid $\eta = 12$. For this set of constants, the vector of the magnetic field has the form is

$$\mathbf{B}_0 = \frac{F_{\mathrm{ell}}\left(\frac{\lambda}{7}, \frac{7}{10}\right)}{\sqrt{(\theta^2 - \lambda^2)(\eta^2 - \theta^2)}}\,\mathbf{e}_\theta - \frac{F_{\mathrm{ell}}\left(\sqrt{\frac{100-\theta^2}{51}}, \sqrt{\frac{51}{100}}\right)}{\sqrt{(\eta^2 - \lambda^2)(\theta^2 - \lambda^2)}}\,\mathbf{e}_\lambda\,. \tag{32}$$

This "vacuum" (gradient) magnetic field is used to produce non-trivial dynamic isotropic and anisotropic plasma equilibria, as shown below.

We remark that though the magnetic field lines of the field (32) have a plane of symmetry $x = 0$, a non-zero choice of constants (A_1, A_2, B_1, B_2) would produce a *completely non-symmetric* magnetic field tangent to a family of ellipsoids.

i. Isotropic dynamic plasma equilibria with ellipsoidal magnetic surfaces. The above vacuum magnetic field $\mathbf{B_0}$ is indeed a trivial solution to the general isotropic plasma equilibrium system (10), (11) with $\mathbf{V} = 0$, $P = P_0 = \mathrm{const}$ and an arbitrary density function $\rho = \rho_0(\mathbf{r})$.

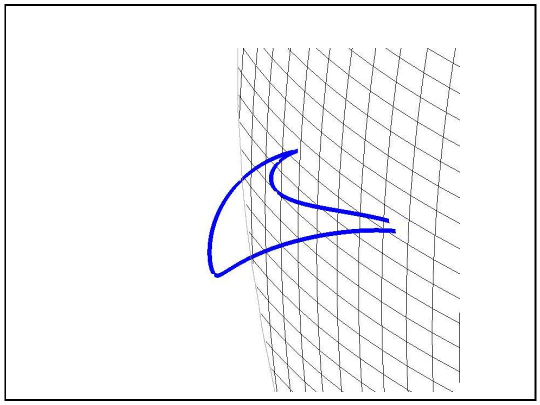

Fig. 2. Magnetic flux tubes in the model of solar flares. The magnetic field (32) in the half-space $y > 0$ can be restricted to a set of one or more disjoint magnetic flux tubes bounded by surface currents. Such force-free configuration, after the application of symmetries (18) and MHD→CGL transformations (22), gives rise to isotropic and anisotropic models of plasma behaviour in arcade solar flares.

The plasma domain $\mathcal{D}$ is either a region in the half–space $y > 0$ between two ellipsoid shells η_1, η_2 : $c < \eta_1 < \eta < \eta_2$ $(c = 10)$, or a set of *flux tubes*, connected sets of adjacent magnetic field lines. (Two sample flux tubes are shown on Fig. 2.) Outside of $\mathcal{D}$ we choose $\rho_0 = 0$ and $\mathbf{B}_0 = 0$ employing a surface current (31).

If we choose $\rho_0(\mathbf{r})$ to be constant on magnetic field lines (plasma streamlines do not exist as there is no flow), then the infinite–parameter transformations (18) become applicable to such configuration. Applying them formally, we obtain a family of isotropic plasma equilibria

$$\begin{gathered}\mathbf{B}_1 = m(\mathbf{r})\mathbf{B}_0\,, \quad \mathbf{V}_1 = \frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\,\mathbf{B}_0\,,\\ \rho_1 = a^2(\mathbf{r})\rho_0(\mathbf{r})\,, \quad P_1 = CP_0 - n^2(\mathbf{r})\,\frac{\mathbf{B}_0^2}{2\mu}\,,\\ m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \mathrm{const}\,,\end{gathered} \tag{33}$$

where $a(\mathbf{r})$, $m(\mathbf{r})$, $n(\mathbf{r})$, $\rho_0(\mathbf{r})$ are functions constant on magnetic field lines and streamlines (which coincide in this case, as $\mathbf{V}_1$ and $\mathbf{B}_1$ are collinear).

The magnetic field lines in the chosen region are not dense on any 2D surface or in any 3D domain, therefore the arbitrary functions $a(\mathbf{r})$, $m(\mathbf{r})$, $n(\mathbf{r})$, $\rho_0(\mathbf{r})$ can be chosen (in a smooth way) to have a constant value on each magnetic field line, thus being in fact functions of *two variables* enumerating all magnetic field lines in the region of interest (for example, η and λ, which specify the beginning of every magnetic field line).

We remark that unlike the initial field $\mathbf{B}_0$, the vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$ are neither potential nor force–free: $\operatorname{curl}\mathbf{B}_1 = \operatorname{grad} m(\mathbf{r}) \times \mathbf{B}_0$ not parallel to $\mathbf{B}_1$. But both $\mathbf{B}_1$ and $\mathbf{V}_1$ satisfy the solenoidality requirement.

Direct verification shows that, with a non-singular choice of the arbitrary functions, the total magnetic energy $E_m = \frac{1}{2}\int_V B_1^2\,\mathrm{d}v$ and the kinetic energy $E_k = \frac{1}{2}\int_V \rho V_1^2\,\mathrm{d}v$ are finite. The magnetic field, velocity, pressure and density $\mathbf{B}_1$, $\mathbf{V}_1$, ρ_1, P_1 are defined in a continuous and differentiable way.

ii. Anisotropic plasma equilibria with ellipsoidal magnetic surfaces. When the mean free path for particle collisions is long compared to Larmor radius, (e.g. in strongly magnetized plasmas), the tensor–pressure CGL approximation should be used. The model suggested here describes a rarefied plasma behaviour in a strong magnetic field looping out of the star surface.

To construct an anisotropic CGL extension of the above isotropic model, we use the MHD→CGL transformations (22). Given $\mathbf{B}_1$, $\mathbf{V}_1$, P_1, ρ_1 determined by (33) with some choice of the arbitrary functions $a(\mathbf{r})$, $m(\mathbf{r})$, $n(\mathbf{r})$, $\rho_0(\mathbf{r})$, we obtain an anisotropic equilibrium $\mathbf{B}_2$, $\mathbf{V}_2$, $p_{\|2}$, $p_{\perp 2}$, ρ_2 defined as

$$\begin{gathered}
\mathbf{B}_2 = f(\mathbf{r})\mathbf{B}_1\,, \quad \mathbf{V}_2 = g(\mathbf{r})\mathbf{V}_1\,, \quad \rho_2 = C_0\frac{\rho_1\mu}{g(\mathbf{r})^2}\,,\\
p_{\perp 2} = C_0\mu P_1 + C_1 + \left(C_0 - \frac{f(\mathbf{r})^2}{\mu}\right)\frac{\mathbf{B}_1^2}{2}\,,\\
p_{\|2} = C_0\mu P_1 + C_1 - \left(C_0 - \frac{f(\mathbf{r})^2}{\mu}\right)\frac{\mathbf{B}_1^2}{2}\,,
\end{gathered} \tag{34}$$

$f(\mathbf{r})$, $g(\mathbf{r})$ are arbitrary functions constant on the magnetic field lines and streamlines, i.e. again on constant on every plasma magnetic field line, and C_0, C_1 are arbitrary constants.

Setting $P_0 = 0$ in (33) and making an explicit substitution, we get

$$\begin{aligned}
\mathbf{B}_2 &= f(\mathbf{r})m(\mathbf{r})\mathbf{B}_0\,,\\
\mathbf{V}_2 &= g(\mathbf{r})\,\frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\,\mathbf{B}_0\,,\\
\rho_2 &= C_0 a^2(\mathbf{r})\,\frac{\rho_0(\mathbf{r})\mu}{g(\mathbf{r})^2}\,,\\
p_{\perp 2} &= C_1 + \frac{\mathbf{B}_0^2}{2\mu}\left(C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r})\right)\,,\\
p_{\|2} &= C_1 + \frac{\mathbf{B}_0^2}{2\mu}\left(f^2(\mathbf{r})m^2(\mathbf{r}) - C_0 C\mu - 2C_0 n^2(\mathbf{r})\right)\,.
\end{aligned} \tag{35}$$

It is known [26, 27] that for the new equilibrium to be free from a fire–hose instability, the transformations (22) must have $C_0 > 0$.

$p_\perp$ is the pressure component perpendicular to magnetic field lines. It is due to the rotation of particles in the magnetic field. Therefore in strongly magnetized or rarified plasmas, where the CGL equilibrium model is applicable, it is natural that the behaviour of $p_\perp$ is connected with that of $\mathbf{B}^2$.

In the observational studies of the solar wind flow in the Earth magnetosheath [40], the relation

$$\frac{p_\perp}{p_\|} = 1 + 0.847\,\frac{B^2}{2p_\|} \tag{36}$$

was proposed. We denote $k(\mathbf{r}) = C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r})$ and select the constants and functions C_0, C, $f(\mathbf{r})$, $m(\mathbf{r})$ so that $k(\mathbf{r}) \geq 0$ in the space region under consideration. From (35), we have:

$$p_{\perp 2} - p_{\| 2} = \frac{\mathbf{B}_0^2}{2\mu}\left(2k(\mathbf{r}) + 2C_0 n^2(\mathbf{r})\right),$$

or

$$\frac{p_{\perp 2}}{p_{\| 2}} = 1 + \frac{2k(\mathbf{r}) + 2C_0 n^2(\mathbf{r})}{\mu f^2(\mathbf{r}) m^2(\mathbf{r})}\,\frac{\mathbf{B}_2^2}{2p_{\| 2}},$$

which generalizes and includes the experimental result (36).

iii. A model of plasma behaviour in arcade solar flares. Solar flares are phenomena that take place in the photospheric region of the solar atmosphere and are connected with a sudden release of huge energies (typically $10^{22} - 10^{25}$ J) ([1], pp. 331 − 348). Particle velocities connected with this phenomenon (about 10^3 m/s) are rather small compared to typical coronal velocities ($\sim 5 \cdot 10^5$ m/s), therefore equilibrium models are applicable.

Morphologically two types of solar flares are distinguished: loop arcades (magnetic flux tubes) and two–ribbon flares. Flares themselves and post–flare loops are grounded in from active photospheric regions.

As noted in [1], p. 332, *"rigorous theoretical modelling has mainly been restricted to symmetric configurations, cylindrical models of coronal loops and two–dimensional arcades."*

The configurations described above can serve as *non-symmetric* 3D isotropic and anisotropic models of quasi–equilibrium plasma in flare and post–flare loops, where magnetic field and inertia terms prevail upon the gravitation potential term in the plasma equilibrium equations:

$$\mathbf{V} \times \operatorname{curl}\mathbf{V} \gg \operatorname{grad}\varphi\,, \quad \frac{1}{\mu}\mathbf{B} \times \operatorname{curl}\mathbf{B} \gg \rho\operatorname{grad}\varphi\,.$$

where φ is the star gravitation field potential.

The characteristic shape of the magnetic field energy density $\mathbf{B}$ and the pressure P along a particular magnetic field line, for the isotropic case, are given on Fig. 3.

Magnetic field lines in the model are not closed; therefore by introducing a surface current of the type (31), a plasma domain $\mathcal{D}$ can indeed be selected to have any tubular loop shape (e.g. the one shown on Fig. 2), and the magnetic

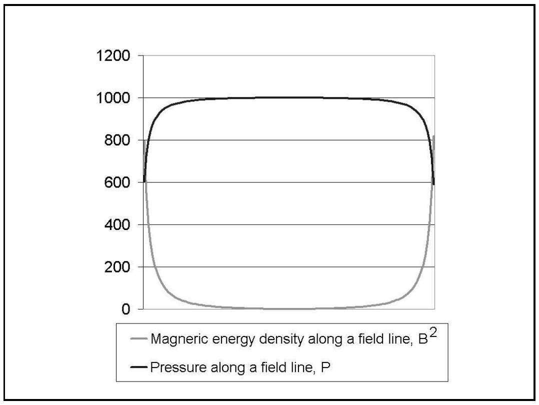

Fig. 3. A model of a solar flare — a coronal plasma loop near an active photospheric region. The figure shows the characteristic shape of the magnetic field energy density $\mathbf{B}^2$ and the pressure P curves along a particular magnetic field line. (The isotropic case.)

field can be assumed zero outside (together with the velocity in dynamic models.) The current sheet introduction is not artificial — as argued in [2], in a general 3D coronal configurations the current sheets between flux tubes are formed (see also: [1], p. 343.)

The above-described isotropic MHD model is valid when the mean free path of plasma particles is much less than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions.

However, the CGL framework must be adopted when plasma is rarefied or strongly magnetized. For such plasmas, we propose the above anisotropic model (35), for which the requirement of plasma being rarefied can be satisfied by choosing $a(\mathbf{r})$ sufficiently small.

4 Conclusion

Isotropic (MHD) the anisotropic (CGL) magnetohydrodynamics equations and their equilibrium reductions (10), (11), (15), (16) are extensively used in the modelling of astrophysical and terrestrial plasmas.

Both MHD and CGL equilibrium systems are essentially non-linear systems of partial differential equations depending on three spatial variables. Therefore the problem of the construction of exact solutions, especially those having natural physical behaviour and relevant for modelling, presents a significant difficulty. Only several types of particular exact solutions have been known (see [27].)

Infinite–parameter Lie groups of symmetries (18) and (20) of ideal incompressible MHD and CGL equilibrium systems, together with their properties and applications in the construction of families of new exact solutions, are discussed in

Section 2. It is important that both of these rich groups symmetries are equivalent to certain Lie point transformations of the corresponding systems, and can be found from the direct Lie group analysis procedure.

Infinite–dimensional transformations between solutions of MHD and CGL equilibrium systems provide another means of the construction of families of physically meaningful anisotropic equilibrium configurations from known isotropic ones (Section 2.3.)

The above symmetries and transformations, however, have to be applied to some known MHD equilibria. They can not change some intrinsic qualitative properties of initial equilibrium solutions, such as solution topology. Therefore, to acheive the diversity of equilibrium models, a demand exists on new particular exact solutions with topologies corresponding to particular applications.

In Section 3, we present a method of building exact 3D isotropic and anisotropic plasma equilibria with and without dynamics, and often without geometrical symmetry, in different geometries and with physically relevant properties.

The representation of the system of static classical plasma equilibrium equations (13) in coordinates connected with magnetic surfaces is formulated in Theorem 2. In such coordinates, the static plasma equilibrium system is reduced to two partial differential equations (24), (25) for two unknown functions. Solutions to these equations in different geometries can be obtained. Special attention is paid to the construction of "vacuum" gradient magnetic fields (Theorem 3.)

Solutions to the system (24), (25), including "vacuum"–type solutions, by virtue of symmetries and transformations discussed in Section 2, give rise to *families* of more complicated dynamic and static, isotropic and anisotropic equilibrium configurations.

A particular example illustrates the use of the introduced coordinate approach. In Section 3.2, exact dynamic isotropic and anisotropic plasma equilibrium configurations that model *solar flares* in the star coronal plasma near an active region are constructed. The resulting model is essentially non-symmetric and is presented in an exact and explicit form; it reproduces important features of solar flares known from observations. The presented equilibrium has ellipsoidal magnetic surfaces.

Other examples of the use of the magnetic surface coordinate representation method can be found in the Thesis [27].

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