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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

APPLICATION OF THE FOURIER SERIES FOR PARTICLE DYNAMICS SIMULATION IN THE PERIODIC MAGNETIC FIELDS

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Given methods emerged in the solution of synchrotron radiation problem in cyclic accelerators [1]. It was necessary to find a set of differential equations for entire closed orbit instead of description for separate sections. The best approach turned out to be an expansion of the magnetic field gradient or components in the Fourier series. To solve the resulting differential equations with periodic coefficients, the averaging theory had to be extended to the third and fourth orders of accuracy. In addition, an original procedure was found within the perturbation theory, which yielded the same results.

Представленные методы используются в решении задачи синхротронного излучения в циклических ускорителях [1]. Вместо описания отдельных секций необходимо было найти систему дифференциальных уравнений для целой замкнутой орбиты. Лучшим подходом оказалось разложение градиента или компонентов магнитного поля в ряд Фурье. Для решения полученного дифференциального уравнения с периодическими коэффициентами теория усреднения должна была достигнуть третьего или четвертого порядка точности. К тому же в рамках теории возмущений была найдена исходная процедура, которая дала те же результаты.

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1. DERIVATION OF EQUATIONS

Without the loss of generality let us, at first, consider a specific example, which will be supplemented by general methods. In our model, an electron rotates in a periodic magnetic field $H = br^{-n}$ (b is a constant, n is the field gradient) along a closed orbit of N elements, where an element consists successively of a focusing arc section of length a and field gradient n_1, a straight section of length l_1, a defocusing magnet with gradient n_2, and again of a free section of length l_2. This is one of the versions of the $FODO$ model, where O stands for the field-free section.

If R is the radius of magnet curvature, then $a = \pi R/N$, and the length of the entire orbit is

$$S = 2\pi R + Nl_1 + Nl_2 = 2\pi R_0,$$

where R_0 is the so-called mean radius. Let us assume that the ratio of free run lengths to the magnet lengths $k = (l_1 + l_2)/2a$ is a small parameter. Then, $R_0 = R(1 + k)$, and the period of the system, taken in terms of the azimuth angle φ, will be defined as $T = 2\pi/N$.

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The magnetic field gradient $n(\varphi)$ can be regarded as a staircase function with discontinuities of the first kind. Expanding $n(\varphi)$ in a Fourier series, we obtain

$$n(\tau) = \frac{a(n_1 - n_2)}{L} + \frac{2}{\pi} \sum_{\nu=1}^{\infty} g_\nu (n_1 \cos \nu\tau_1 - n_2 \cos \nu\tau_2), \tag{1}$$

where $\tau = N\varphi$,

$$g_\nu = \sin \pi\nu \, \frac{a}{L} \Big/ \nu, \quad L = 2a + l_1 + l_2, \quad \tau_1 = \tau - \frac{\pi a}{L}, \quad \tau_2 = \tau - \pi \frac{3a + 2l_1}{L}.$$

It would be more justified to go from a series to a partial sum in (1), because edge effects may arise at the boundary of the magnet, and switching from n_1 to n_2 is smoother. For example, at $\varphi = 0$ we get, by the Dirichlet theorem, that $n(\tau) = n_1/2$; at $\varphi = (3a/2 + l_1)/L(2\pi/N)$ (the middle of the second magnet) $n(\tau) = -n_2$, and so on.

Finally, the Hill equations of small betatron oscillations in the linear approximation take the form

$$\frac{d^2\rho}{d\tau^2} + \frac{1}{N^2}(1 - (1+k)^2 n(\tau))\rho = 0, \tag{2}$$

$$\frac{d^2 z}{d\tau^2} + \frac{(1+k)^2}{N^2} n(\tau) z = 0, \tag{3}$$

where $\rho = r - R_0$.

As to the main orbital motion of the particle, it has some features resulting from switching back and forth between rotation and rectilinear motion. Since only vertical oscillations affect the radiation properties, we made some simplifications. We averaged the leading magnetic field H_0 over the entire period of the system and assumed that the mean radius of the electron rotation is R_0.

Then, the angular velocity can be represented as

$$\dot{\varphi} = \frac{\omega_0}{1+k} \left[1 - \frac{\rho}{R_0} + \frac{3}{2} \frac{\rho^2}{R_0^2} + \int n(\varphi) \left(\frac{z\dot{z}}{R^2} - \frac{\rho\dot{\rho}}{R^2} \right) dt \right], \tag{4}$$

where $\omega_0 = ceH_0/E$.

2. BOGOLIUBOV–MITROPOLSKY METHOD

Following the averaging theory [2], we represent Eq. (3) in the standard form

$$\frac{dZ}{d\tau} = \varepsilon \cdot G \cdot Z, \tag{5}$$

where the components of the vector Z are z and $(1/\varepsilon)\, dz/d\tau$. Here, we have

$$G = \begin{pmatrix} 0 & 1 \\ -g(\tau) & 0 \end{pmatrix}, \quad g(\tau) = (1+k)^2 n(\tau),$$

and $\varepsilon = 1/N$ is the small parameter.

We can consider a more general matrix equation of the form

$$\frac{dZ}{d\tau} = \varepsilon \cdot Y(\tau, Z).$$

For the function $Y(\tau, Z)$ the operation of averaging is defined as

$$\langle Y(\tau, Z)\rangle = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T Y(\tau, Z)\, d\tau, \tag{6}$$

and the integrating operator

$$\widetilde{Y}(\tau, \xi) = \int [Y(\tau, \xi) - \langle Y(\tau, \xi)\rangle]\, d\tau$$

is introduced. The vector ξ is found from the system

$$\frac{d\xi}{d\tau} = \varepsilon \langle Y(\tau, \xi)\rangle,$$

and the first approximation is defined as

$$Z(\tau) = \xi(\tau) + \varepsilon \widetilde{Y}(\tau, \xi).$$

Linear system (5) turns out to be the Caratheodory differential equation system [3, 4]. Indeed, matrix $G(\tau)$ is defined and is continuous at almost all τ, except those points making up a set of measure zero. Moreover, it is seen to be limited. Thus, summation of the matrix elements is possible on each section from the interval $(0, 2\pi)$. Then, by Theorem 3 from Ch. 1 in [4], the solution of (5) with any initial condition $Z(\tau_0) = Z_0$ exists in the entire interval $(0, 2\pi)$. Since here

$$|Y(\tau, Z_2) - Y(\tau, Z_1)| \leqslant |G| \cdot |Z_2 - Z_1|,$$

i.e., the Lipschitz condition is fulfilled, this solution will be the only one. That is why an arbitrary choice of the initial point can be used in calculating the trajectory by the matrix method, where the matrices of each structure are multiplied and a return to the initial point occurs.

Now, let us go back to the general method, which was written for the second approximation in [5]. With the iteration procedure, where a new correction is successively added, any approximation can be described.

If we take the first four orders, we obtain

$$Z(\tau) = \xi + \sum_{i=1}^{4} \varepsilon^i Y_i, \tag{7}$$

where

$$Y_1 = \widetilde{Y}, \quad Y_2 = \frac{\partial Y}{\partial \xi} Y_1 - \int \frac{\partial Y_1}{\partial \xi} d\tau \langle Y\rangle,$$

$$Y_3 = \frac{\partial Y}{\partial \xi} Y_2 - \int \frac{\partial Y_1}{\partial \xi} d\tau \left\langle \frac{\partial Y}{\partial \xi} Y_1 \right\rangle - \int \frac{\partial Y_2}{\partial \xi} d\tau \langle Y\rangle,$$

$$Y_4 = \frac{\partial Y}{\partial \xi} Y_3 - \int \frac{\partial Y_1}{\partial \xi} d\tau \left\langle \frac{\partial Y}{\partial \xi} Y_2 \right\rangle - \int \frac{\partial Y_2}{\partial \xi} d\tau \left\langle \frac{\partial Y}{\partial \xi} Y_1 \right\rangle - \int \frac{\partial Y_3}{\partial \xi} d\tau \langle Y\rangle.$$

According to [2], the error of this approximation is ε^5. Now, let us set $Y(\tau, Z) = G \cdot Z$. Then, from (7) we find that

$$Z(\tau) = \left(1 + \sum_{i=1}^{4} \varepsilon^i G_i\right) \xi, \tag{8}$$

where

$$\frac{d\xi}{d\tau} = \varepsilon \left\langle G \left(1 + \sum_{i=1}^{4} \varepsilon^i G_i\right)\right\rangle \xi. \tag{9}$$

Differentiation with respect to the vector ξ can be illustrated by the following example:

$$\frac{\partial}{\partial \xi} Y(\tau, \xi) = \begin{pmatrix} \dfrac{\partial}{\partial \xi_1}(g_{11}\xi_1 + g_{12}\xi_2) & \dfrac{\partial}{\partial \xi_2}(g_{11}\xi_1 + g_{12}\xi_2) \\ \dfrac{\partial}{\partial \xi_1}(g_{21}\xi_1 + g_{22}\xi_2) & \dfrac{\partial}{\partial \xi_2}(g_{21}\xi_1 + g_{22}\xi_2) \end{pmatrix} = G.$$

The matrices G_i then have the form

$$G_1 = \widetilde{G}, \quad G_2 = \widetilde{GG_1} - \widetilde{G_1}\langle G\rangle,$$
$$G_3 = \widetilde{GG_2} - \widetilde{G_1}\langle GG_1\rangle - \widetilde{G_2}\langle G\rangle,$$
$$G_4 = \widetilde{GG_3} - \widetilde{G_1}\langle GG_2\rangle - \widetilde{G_2}\langle GG_1\rangle - \widetilde{G_3}\langle G\rangle.$$

Expressed in terms of the function $g(\tau)$, they are

$$G_1 = \begin{pmatrix} 0 & 0 \\ -\widetilde{g} & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -\widetilde{\widetilde{g}} & 0 \\ 0 & \widetilde{\widetilde{g}} \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 2\widetilde{\widetilde{\widetilde{g}}} \\ \widetilde{g\widetilde{\widetilde{g}}} + \gamma_{z1}^2 \widetilde{\widetilde{\widetilde{g}}} & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} \widetilde{\widetilde{g\widetilde{\widetilde{g}}}} + 3\gamma_{z1}^2 \widetilde{\widetilde{\widetilde{\widetilde{g}}}} & 0 \\ 0 & -2\widetilde{g\widetilde{\widetilde{\widetilde{g}}}} - \widetilde{\widetilde{g\widetilde{\widetilde{g}}}} - \gamma_{z1}^2 \widetilde{\widetilde{\widetilde{\widetilde{g}}}} \end{pmatrix}.$$

Here, $\gamma_{z1}^2 = (n_1 - n_2)(1 + k)/2$,

$$g(\tau) = \gamma_{z1}^2 + \frac{2(1+k)^2}{\pi} \sum_{\nu=1}^{\infty} g_\nu (n_1 \cos \nu\tau_1 - n_2 \cos \nu\tau_2),$$
$$\langle g(\tau)\rangle = \gamma_{z1}^2, \quad \widetilde{g(\tau)} = \frac{2(1+k)^2}{\pi} \sum_{\nu=1}^{\infty} \frac{g_\nu}{\nu} (n_1 \sin \nu\tau_1 - n_2 \sin \nu\tau_2),$$

and so on. From (9) we get

$$\frac{d^2\xi}{d\tau^2} = -\frac{1}{N^2}\left[\gamma_{z1}^2 - \frac{1}{N^2}\left\langle g\widetilde{\widetilde{g}}\right\rangle + \frac{1}{N^4}\left\langle g\widetilde{\widetilde{g\widetilde{\widetilde{g}}}} + 3\gamma_{z1}^2 g\widetilde{\widetilde{\widetilde{\widetilde{g}}}}\right\rangle\right] \cdot \xi.$$

Let us denote the bracketed term in the last expression by ν_z^2. Then, $\xi = B\cos\left(\frac{\nu_z}{N}\tau + \psi\right)$, where B and ψ are arbitrary constants, and the frequency squared is

$$\nu_z^2 = \frac{n_1 - n_2}{2}(1+k) + \frac{\pi^2}{N^2}\frac{N_1}{48} + \frac{\pi^4}{N^4}\frac{(n_1 - n_2)N_2}{5 \cdot 48(1+k)}, \tag{10}$$

where

$$\begin{aligned} N_1 &= (n_1^2 + n_2^2)(1+2k)^2 + 2n_1 n_2(1 + 4k + 4k_1 k_2 - k_1^2 - k_2^2), \\ N_2 &= (n_1 + n_2)^2\left(1 + 6k + \frac{2}{3}k^2(19 + 16k + 4k^2)\right) + n_1 n_2 N_3, \\ N_3 &= -\frac{32}{3}k^2(1 + 4k + k^2) + 10k_1 k_2(1 + 4k + k_1 k_2), \end{aligned}$$

and $k_1 = l_1/a, k_2 = l_2/a$. Detailed formula (10) illustrates the effectiveness of the method.

Substituting G_i into matrix equality (8), we find the solution of initial equation (3) in the form

$$z = \left[1 - \frac{1}{N^2}\tilde{\tilde{g}} + \frac{1}{N^4}\left(\widetilde{g\tilde{\tilde{g}}} + 3\gamma_{z1}^2\tilde{\tilde{\tilde{g}}}\right)\right] \cdot \xi + \frac{2}{N^3}\tilde{\tilde{g}}\frac{d\xi}{d\varphi}.$$

Eventually, the asymptotics up to $1/N^3$ are written as

$$z = B\cos\left(\frac{\nu_z}{N}\tau + \psi\right)(1 + S_1 + \gamma_{z1}^2 S_2) + B\sin\left(\frac{\nu_z}{N}\tau + \psi\right)\gamma_{z1}S_3, \tag{11}$$

where

$$\begin{aligned} S_1 &= \frac{2(1+k)^2}{\pi N^2}\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^2}(n_1\cos\nu\tau_1 - n_2\cos\nu\tau_2), \\ S_2 &= \frac{8(1+k)^2}{\pi N^4}\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^4}(n_1\cos\nu\tau_1 - n_2\cos\nu\tau_2), \\ S_3 &= \frac{4(1+k)^2}{\pi N^3}\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^3}(n_1\sin\nu\tau_1 - n_2\sin\nu\tau_2). \end{aligned}$$

Now, we turn to Eq. (2), which has its own specific features. In particular, $\gamma_{\rho 1} = 1 - \gamma_{z1}^2$,

$$G = \begin{pmatrix} 0 & 1 \\ -1 + g & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 0 \\ \tilde{g} & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} \tilde{\tilde{g}} & 0 \\ 0 & -\tilde{\tilde{g}} \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & -2\tilde{\tilde{\tilde{g}}} \\ \widetilde{g\tilde{\tilde{g}}} - (1 + \gamma_{\rho 1}^2)\tilde{\tilde{\tilde{g}}} & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} \widetilde{g\tilde{\tilde{\tilde{g}}}} - (1 + 3\gamma_{\rho 1}^2)\tilde{\tilde{\tilde{\tilde{g}}}} & 0 \\ 0 & (3 + \gamma_{\rho 1}^2)\tilde{\tilde{\tilde{\tilde{g}}}} - 2g\tilde{\tilde{\tilde{\tilde{g}}}} - \widetilde{\widetilde{g\tilde{\tilde{g}}}} \end{pmatrix}.$$

The equation for ξ has the form

$$\frac{d^2\xi}{d\tau^2} + \frac{\nu_\rho^2}{N^2}\xi = 0,$$

where

$$\xi = A\cos\left(\frac{\nu_\rho}{N}\tau + \chi\right),$$

and the frequency squared is

$$\nu_\rho^2 = 1 - \frac{n_1 - n_2}{2}(1+k) + \frac{\pi^2}{N^2}\frac{N_1}{48} + \frac{\pi^4}{N^4}\frac{(2-(1+k)(n_1-n_2))N_2}{5\cdot 48(1+k)^2}. \tag{12}$$

In the last form, the solution is as follows:

$$\rho = A\cos\left(\frac{\nu_\rho}{N}\tau + \chi\right)(1 - S_1 - \gamma_{\rho 1}^2 S_2) - A\sin\left(\frac{\nu_\rho}{N}\tau + \chi\right)\gamma_{\rho 1} S_3. \tag{13}$$

Averaging over fast oscillations leaves only the first terms in asymptotic expressions (11) and (13). Consequently, A and B can be interpreted as the amplitudes of the main cosine oscillations and χ and ψ — as their initial phases. Additional terms show how is complicated this particle motion.

There are no divergent hyperbolic functions in (11) and (13), which complies with the strong focusing principle resulting in the stability of the motion as a whole.

The series that we have here can be expressed either in terms of the Bernoulli polynomials $B_i(x)$ or those that converge so fast that can be found numerically. For example, in the first case,

$$\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^2}(n_1\cos\nu\tau_1 - n_2\cos\nu\tau_2) = \frac{\pi^3}{3}\left[n_1\left(B_3\left(\frac{\tau}{2\pi}\right) - B_3\left(\frac{\tau}{2\pi} - \frac{a}{L}\right)\right) + \right.$$
$$\left. + n_2\left(B_3\left(\frac{\tau}{2\pi} - \frac{2a+l_1}{L}\right) - B_3\left(\frac{\tau}{2\pi} - \frac{a+l_1}{L}\right)\right)\right].$$

Here, $x \in [0, 2\pi]$ for the arguments of the polynomials. If this condition is not fulfilled, one must go to another period. Thus, there is no doubt about the convergence of individual terms in the asymptotics. To study this matter in general, we must again turn to (5), where $G \cdot Z = Y(\tau, Z)$.

The vector function $Y(\tau, Z)$ satisfies the Caratheodory conditions for the existence of a continuous solution, because it is τ-measurable at each fixed Z, and Z-continuous at each fixed τ. For this function and its averaged value

$$\langle Y(\tau,\xi)\rangle = \langle G\rangle \cdot \xi$$

the Lipschitz condition is fulfilled. In addition, it can be shown that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \widetilde{Y}(\tau, Z)\,d\tau = 0.$$

Thus, all conditions of Theorem 1.1 from [6] are fulfilled, which indicates the proximity of solution z and ξ for the initial and averaged systems.

3. PERTURBATION THEORY TECHNIQUE

Now, let us obtain expression (11) in a different way. We look for the solution of (3) in the form

$$z = \exp(i\gamma_z \tau)\varphi_z(\tau),$$

where $\varphi_z(\tau)$ is a periodic function. Since we are interested in motion only inside the stable region called the "necktie" of stability, we shall assume that $\operatorname{Im}\gamma_z = 0$. A similar condition should be set for γ_ρ as well. This approach is close to the Whittaker method and the method of stretched parameters for the Mathieu equation in [7].

For the function $\varphi_z(\tau)$ we get a new differential equation

$$\frac{d^2\varphi_z}{d\tau^2} + 2i\gamma_z \frac{d\varphi_z}{d\tau} + \left[\frac{(1+k)^2 n(\tau)}{N^2} - \gamma_z^2\right]\varphi_z = 0,$$

its solutions being defined as the asymptotic series

$$\varphi_z(\tau) = \varphi_0(\tau) + \sum_{i=1}^{\infty} \frac{\varphi_i(\tau)}{N^i}, \quad \gamma_z = \sum_{i=1}^{\infty} \frac{\gamma_{zi}}{N^i}.$$

The small parameter is still $1/N$. Setting coefficients at the same powers of the parameter equal to zero, we arrive at the following chain of equations:

$$\ddot{\varphi}_0 = 0, \quad \ddot{\varphi}_1 + 2i\gamma_{z1}\dot{\varphi}_0 = 0,$$
$$\ddot{\varphi}_2 + 2i(\gamma_{z1}\dot{\varphi}_1 + \gamma_{z2}\dot{\varphi}_0) + ((1+k)^2 n(\tau) - \gamma_{z1}^2)\varphi_0 = 0,$$
$$\ddot{\varphi}_3 + 2i(\gamma_{z1}\dot{\varphi}_2 + \gamma_{z2}\dot{\varphi}_1 + \gamma_{z3}\dot{\varphi}_0) + ((1+k)^2 n(\tau) - \gamma_{z1}^2)\varphi_1 - 2\gamma_{z1}\gamma_{z2}\varphi_0 = 0, \quad \text{and so on.}$$

According to [2, 7], the secular terms must be eliminated in the solutions for these equations. In the averaging theory, this corresponds to the existence of the limit on the right-hand side in (6).

Then, from the first two equations, we get $\varphi_0 = b$, $\varphi_1 = b_1$, where b, b_1 are constants. From the third expression the equality $\gamma_{z1}^2 = (n_1 - n_2)(1+k)/2$ follows. The solution itself is written as

$$\varphi_2 = b\frac{2(1+k)^2}{\pi} \sum_{\nu=1}^{\infty} \frac{g_\nu}{\nu^2}(n_1 \cos\nu\tau_1 - n_2 \cos\nu\tau_2).$$

Then, we subsequently find $\gamma_{z2} = 0$,

$$\varphi_3 = -i\gamma_{z1} b\frac{4(1+k)^2}{\pi} \sum_{\nu=1}^{\infty} \frac{g_\nu}{\nu^3}(n_1 \sin\nu\tau_1 - n_2 \sin\nu\tau_2) + \frac{b_1}{b}\varphi_2, \quad \gamma_{z3} = \frac{\pi^2 N_1}{96\gamma_{z1}},$$
$$\varphi_4 = \gamma_{z1}^2 b N^2 S_2 - i\gamma_{z1} b_1 N^3 S_4 + bN^4 S_3, \quad \gamma_{z4} = 0, \quad \text{and so on.}$$

The square of the frequency $\nu_z = N\gamma_z$ is defined as

$$\nu_z^2 = \gamma_{z1}^2 + 2\gamma_{z1}\gamma_{z3}/N^2.$$

Substituting γ_{z1} and γ_{z3}, we get formula (10) up to $1/N^2$.

The general solution can be built as

$$z = C\exp(i\gamma_z\tau)\varphi_z(\tau) + C^*\exp(-i\gamma_z\tau)\varphi_z^*(\tau).$$

Redesignating the arbitrary constants as

$$C\left(b + \frac{b_1}{N}\right) = \left(\frac{B}{2}\right)\exp(i\psi),$$

we again arrive at expression (11).

Frequency (12) and asymptotic (13) can be derived in a similar manner, if the solution of (2) is sought in the form $\rho = \exp(i\gamma_\rho\tau)\varphi_\rho(\tau)$.

It turns out that the expansion parameter in (11) and (13) is n/N^2. For example, for the former DESY synchrotron it was equal to 0.15. The next-order correction will be about 2%.

Averaging the total velocity squared over period, we have

$$\langle v^2\rangle = R_0^2\left(\frac{\omega_0}{1+k}\right)^2\left(1 + \nu_\rho^2\frac{A^2}{R_0^2} + \nu_z^2\frac{B^2}{R_0^2}\right).$$

The angular frequency of revolution ω_0 becomes $1+k$ times smaller due to the effect of the straight sections.

Summing up the last two sections, we can formulate the following theorem.

Theorem. The approximate solutions found for Eqs. (2) and (3) by the averaging method and the perturbation method proposed in this section lead to the same result for all orders.

This can be demonstrated by the induction method.

CONCLUSION

First of all, it should be stressed that the above analysis was carried out for only one period. However, the same conditions occur as the particle passed to another magnetic period and, thus, generalization to the entire closed orbit is possible. Moreover, we were only interested (from the point of view of the long-term use of radiation) in the stable motion of a particle inside the envelope, where it makes thousands of turns. In treating the problem of particle motion in an accelerator, we must point out that the Caratheodory conditions are fulfilled in this case as well, because the set of discontinuity gaps for the gradient is large but denumerable.

Taking into account also [8], it seems probable to consider a more general equation

$$\frac{d^2x}{d\tau^2} + \epsilon^2(a_0 + f(\tau))x = 0$$

in much the same way, where $f(\tau)$ is the periodic function and ϵ is the small parameter, which applies to other physical problems.

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