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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

ON THE THEORY OF WAVE PACKETS

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In this paper we discuss some aspects of the theory of wave packets. We consider a popular non-covariant Gaussian model used in various applications and show that it predicts too slow a longitudinal dispersion rate for relativistic particles. We revise this approach by considering a covariant model of Gaussian wave packets, and examine our results by inspecting a wave packet of an arbitrary form. A general formula for the time dependence of the dispersion of a wave packet of an arbitrary form is found. Finally, we give a transparent interpretation of the disappearance of the wave function over time due to the dispersion — a feature often considered undesirable, but which is unavoidable for wave packets. We find, starting with simple examples, proceeding with their generalizations and finally by considering the continuity equation, that the integral over time of both the flux and probability densities is asymptotically proportional to the factor $1/|\mathbf{x}|^2$ in the rest frame of the wave packet, just as in the case of an ensemble of classical particles.

В данной работе мы обсуждаем некоторые аспекты теории волновых пакетов. Мы рассматриваем популярную нековариантную гауссову модель, используемую в различных приложениях, и показываем, что она предсказывает слишком медленную продольную скорость расплывания для релятивистских частиц. Мы уточняем это приближение рассмотрением ковариантной модели гауссовых волновых пакетов, а также исследуем этот вопрос для волновых пакетов произвольной формы. Получена общая формула зависимости от времени дисперсии волнового пакета произвольной формы. Наконец, мы даем прозрачную интерпретацию уменьшения модуля волновой функции со временем из-за расплывания волнового пакета (непременное свойство волновых пакетов, часто рассматриваемое как нежелательное в разных приложениях). Мы находим, начиная с простых примеров, затем переходя к наиболее общей форме волнового пакета и, наконец, исследуя уравнение непрерывности, что интеграл по времени от плотностей вероятности и тока асимптотически стремится к фактору $1/|\mathbf{x}|^2$ в системе покоя волнового пакета, в точности как для случая ансамбля классических частиц.

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INTRODUCTION

A classical or quantum object composed of multiple waves with trajectory characteristics of a solid body is often called a *wave packet*. A prime example of a quantum wave packet is the wave function describing the free propagation of a particle. Waves in fluids and gases are examples of classical wave packets in our everyday life.

Wave packets are known to spread with the passage of time. In other words, the spatial size of the wave packet grows over time, while its amplitude vanishes. This well-known

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unavoidable feature of wave packets is often considered as their «disadvantage» in various applications.

In particular, while it is impossible to build a consistent scattering theory without wave packets [1, 2] it is not rare to meet an argument that their vanishing with time makes them not quite adequate objects for initial and final states. Indeed, the initial (final) state is defined at infinitely far past (future) moment in time at which any wave packet should vanish, thus making their use in S theory sometimes arguable.

However, as we show in this paper this problem is not critical for the S-matrix formalism, as the spreading of wave packets has a very clear interpretation. The dispersion leads precisely to a $1/|\mathbf{x}|^2$ suppression of the time-integrated probability to observe the particle, just as one would expect for an ensemble of classical «particles», thus leading to transparent normalization factors of the initial and final states.

We will first introduce some definitions and notations used within this paper, and then outline its layout. Throughout the paper we will use the Natural units where $\hbar = c = 1$. For definiteness we will examine a quantum wave packet, while our results will also be valid for classical wave packets because we do not consider particle creation or annihilation in this paper. The wave packet for a spinless particle with mass m has the form

$$|\text{wave packet}\rangle = \int \frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}} |\mathbf{k}\rangle, \tag{1}$$

where $|\mathbf{k}\rangle$ is the Fock state with definite 3-momentum $\mathbf{k}$ and energy $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, and $\phi(\mathbf{k})$ is a Lorentz-invariant function assumed to be narrow around some momentum $\mathbf{p}$ which we will define later on. As we will show in Sec. 2 a real-valued $\phi(\mathbf{k})$ corresponds to the wave packet with mean three-dimensional position:

$$\langle \mathbf{x} \rangle = 0 \text{ at time } t = 0.$$

The action of the translation operator $\mathrm{e}^{i\hat{P}x_0}$ (where $\hat{P}$ is the operator of 4-momentum, and $x_0 = (t_0, \mathbf{x}_0)$ — the four-dimensional displacement vector) onto the wave-packet state in (1):

$$\mathrm{e}^{i\hat{P}x_0} \int \frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}} |\mathbf{k}\rangle = \int \frac{d\mathbf{k}\,\mathrm{e}^{ikx_0}\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}} |\mathbf{k}\rangle, \tag{2}$$

translates the wave-packet state in space and time. Therefore, the wave packet in (1) with

$$\phi(\mathbf{k}) = \mathrm{e}^{ikx_0} |\phi(\mathbf{k})| \tag{3}$$

has the mean position $\langle \mathbf{x} \rangle = \mathbf{x}_0$ at $t = t_0$. Since it essentially gives us no new information, in this paper we will consider the case where $\mathbf{x}_0 = 0$ at $t_0 = 0$, assuming a real-valued $\phi(\mathbf{k})$ function.

In coordinate space the wave packet is characterized by the Lorentz-invariant wave function, which can be obtained by projecting the wave-packet state (1) onto the $\langle 0|\hat{\Psi}(x)$ state, where $\hat{\Psi}(x)$ is the quantum field operator for the (pseudo)scalar particle, as follows:

$$\psi(x) = \langle 0|\hat{\Psi}(x)|\text{wave packet}\rangle = \int \frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}} \mathrm{e}^{-ikx}. \tag{4}$$

It is apparent that $\psi(x)$ satisfies the Klein–Gordon equation. If $\psi(0,\mathbf{x})$ is known, then $\phi(\mathbf{k})$ could be found as follows:

$$\frac{\phi(\mathbf{k})}{2E_{\mathbf{k}}} = \int d\mathbf{x}\,\psi(0,\mathbf{x})\,\mathrm{e}^{-i\mathbf{k}\mathbf{x}}. \tag{5}$$

The 4-vector of the flux density $j_\mu(x) = (\rho(x), \mathbf{j}(x))$ is defined in the usual way for the Klein–Gordon equation:

$$j_\mu(x) = i\left(\psi^*(x)\partial_\mu\psi(x) - \psi(x)\partial_\mu\psi^*(x)\right). \tag{6}$$

The probability density, which is not relativistic invariant, can be normalized in a relativistically invariant way as follows:

$$\int d\mathbf{x}\,\rho(t,\mathbf{x}) = \int \frac{d\mathbf{k}\,|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}} = 1 \tag{7}$$

which corresponds to one particle within the Universe. The spatial integral over the flux density $\mathbf{j}(t,\mathbf{x})$ is equal to the mean velocity of the wave packet, as can be seen from the following:

$$\int d\mathbf{x}\,\mathbf{j}(t,\mathbf{x}) = \int \frac{d\mathbf{k}\,|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}}\mathbf{v}_{\mathbf{k}} = \langle\mathbf{v}\rangle. \tag{8}$$

By definition, the mean energy $\langle \mathrm{E}\rangle$ and mean momentum $\langle \mathbf{P}\rangle$ of the wave-packet state are obtained from

$$\langle \mathrm{P}_\mu\rangle = \frac{\langle \text{wave packet}|\hat{\mathrm{P}}_\mu|\text{wave packet}\rangle}{\langle \text{wave packet}|\text{wave packet}\rangle}, \tag{9}$$

where $\hat{\mathrm{P}}_\mu$ is the μth component of the 4-momentum operator acting on the Fock state as $\hat{\mathrm{P}}_\mu|\mathbf{k}\rangle = k_\mu|\mathbf{k}\rangle$. Therefore,

$$\langle \mathrm{E}\rangle = \int \frac{d\mathbf{k}|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}}E_{\mathbf{k}}, \tag{10}$$

$$\langle \mathbf{P}\rangle = \int \frac{d\mathbf{k}|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}}\mathbf{k}. \tag{11}$$

Equations (1), (7), (8), (10), (11) suggest that $|\phi(\mathbf{k})|^2/2E_{\mathbf{k}}$ is the probability density to have 3-momentum $\mathbf{k}$ for the state in (1). The paper is organized as follows. We begin with a well-known example of a noncovariant Gaussian wave packet in Subsec. 1.1 to illustrate the main features of a dispersive wave packet. We observe that the wave packet disperses in both the longitudinal and transverse directions relative to the mean velocity vector. However, it is shown that the speed of the longitudinal dispersion predicted by the noncovariant model is too slow after examining the covariant Gaussian model of the wave packet given in Subsec. 1.2. For the considered examples we will find that the time-integrated flux density asymptotically follows $1/|\mathbf{x}|^2$. In Sec. 2 we generalize our calculations for wave packets of an arbitrary form restricted by (3). We will find that, on average, a wave packet follows the classical trajectory. We will also produce a general formula for the time dependence of the dispersion of the wave packet. After inspecting this formula we realize that the prediction of the noncovariant Gaussian model for the longitudinal dispersion is indeed not correct for ultrarelativistic particles, while the covariant model agrees with the prediction using more general considerations. In Subsec. 2.2 we examine the asymptotic natures of the time-integrated flux and probability densities, and confirm that a wave packet of an arbitrary form will follow a $1/|\mathbf{x}|^2$ trend. In Sec. 3 we revisit the $1/|\mathbf{x}|^2$ tendencies from another point of view — by considering possible conclusions from the continuity equation, which holds true not only for the Klein–Gordon equation, from which we performed all of our calculations in this paper, but also for other equations like those of Schroedinger and Dirac. Finally, in Sec. 4 we discuss the main results of this paper and draw conclusions.

1. GAUSSIAN WAVE PACKET

1.1. Noncovariant Gaussian Wave Packet. Let us consider, as a useful illustration, a well-known example of a noncovariant Gaussian wave packet with $\phi(\mathbf{k}) = \varphi_{\rm G}(\mathbf{k})$ assumed to be

$$\varphi_{\rm G}(\mathbf{k}) = \sqrt{2E_{\mathbf{p}}}\left(\frac{2\pi}{\sigma_{\rm p}^2}\right)^{3/4} \exp\left[-\frac{(\mathbf{k}-\mathbf{p})^2}{4\sigma_{\rm p}^2}\right], \tag{12}$$

where $\sigma_{\rm p}$ is the *constant* Gaussian width in the momentum distribution of the wave packet. The space–time wave function $\psi(x) = \psi_{\rm G}(x)$ could be obtained from (4) by assuming small enough $\sigma_{\rm p}$ to expand $E_{\mathbf{k}}$ in the exponent around $\mathbf{k} = \mathbf{p}$:

$$E_{\mathbf{k}} = E_{\mathbf{p}} + \mathbf{v}_{\mathbf{p}}(\mathbf{k}-\mathbf{p}) + \frac{m^2}{2E_{\mathbf{p}}^3}(\mathbf{k}-\mathbf{p})^2 + \frac{(\mathbf{p}\times\mathbf{k})^2}{2E_{\mathbf{p}}^3} + \ldots, \tag{13}$$

where $\mathbf{v} = \mathbf{p}/E_{\mathbf{p}}$, and by replacing $E_{\mathbf{k}}$ in the denominator of $\phi(\mathbf{k})/2E_{\mathbf{k}}$ with $E_{\mathbf{p}}$. Then, the remaining Gaussian integrals in (4) yield

$$\psi_{\rm G}(x) = \frac{\exp\left[-ipx - \dfrac{(\mathbf{x}_{\rm L}-\mathbf{v}t)^2}{4\sigma_{\rm x}^2(1+it/\tau_{\rm L})} - \dfrac{\mathbf{x}_{\rm T}^2}{4\sigma_{\rm x}^2(1+it/\tau_{\rm T})}\right]}{(2\pi)^{3/4}\sqrt{2E_{\mathbf{p}}}\sigma_{\rm x}^{3/2}\sqrt{(1+it/\tau_{\rm L})}(1+it/\tau_{\rm T})}, \tag{14}$$

where $p = (E_{\mathbf{p}}, \mathbf{p})$, $\sigma_{\rm x}^2 = 1/4\sigma_{\rm p}^2$ and

$$\tau_{\rm L} = \gamma_{\mathbf{p}}^3\tau, \quad \tau_{\rm T} = \gamma_{\mathbf{p}}\tau, \quad \tau = 2\sigma_{\rm x}^2 m, \quad \gamma_{\mathbf{p}} = \frac{E_{\mathbf{p}}}{m}. \tag{15}$$

$\mathbf{x}_{\rm L}$ and $\mathbf{x}_{\rm T}$ are components of $\mathbf{x}$ parallel and perpendicular, respectively, to the average (and most probable) velocity vector $\mathbf{v}$.

As one might observe, $\psi_{\rm G}(x)$ describes a wave packet which spreads over time. To present it in a more transparent fashion let us examine $|\psi_{\rm G}(x)|$:

$$|\psi_{\rm G}(t,\mathbf{x})| = \frac{\exp\left[-\dfrac{(\mathbf{x}_{\rm L}-\mathbf{v}t)^2}{4\sigma_{\rm x}^2(1+t^2/\tau_{\rm L}^2)} - \dfrac{\mathbf{x}_{\rm T}^2}{4\sigma_{\rm x}^2(1+t^2/\tau_{\rm T}^2)}\right]}{(2\pi)^{3/4}\sqrt{2E_{\mathbf{p}}}\sigma_{\rm x}^{3/2}\left(1+t^2/\tau_{\rm L}^2\right)^{1/4}\sqrt{1+t^2/\tau_{\rm T}^2}}. \tag{16}$$

Considering this example, a noncovariant Gaussian wave packet is described at $t = 0$ by

$$\psi_{\rm G}(0,\mathbf{x}) = \frac{1}{(2\pi)^{3/4}\sqrt{2E_{\mathbf{p}}}\sigma_{\rm x}^{3/2}} \exp\left[i\mathbf{p}\mathbf{x} - \frac{\mathbf{x}^2}{4\sigma_{\rm x}^2}\right] \tag{17}$$

and at later times spreads in both the longitudinal and transverse directions. The squares of the longitudinal ($\sigma_{\rm xL}^2(t)$) and transverse ($\sigma_{\rm xT}^2(t)$) dispersions read

$$\sigma_{\rm xL}^2(t) = \sigma_{\rm x}^2(1 + t^2/\tau_{\rm L}^2), \tag{18}$$

$$\sigma_{\rm xT}^2(t) = \sigma_{\rm x}^2(1 + t^2/\tau_{\rm T}^2), \tag{19}$$

where $\tau_{\rm L}$ and $\tau_{\rm T}$, given by (15), are related to each other by $\tau_{\rm L} = \gamma_{\mathbf{p}}^2\tau_{\rm T}$. In what follows we will refer to $\tau_{\rm L}$ and $\tau_{\rm T}$ as the longitudinal and transverse dispersion times, respectively. τ is the dispersion in the rest frame of the wave packet when, obviously, $\tau_{\rm L} = \tau_{\rm T} = \tau$.

The noncovariant model suggests that the wave packet is expected to disperse more slowly in the longitudinal direction by a factor of $\gamma_{\mathbf{p}}^2$ as compared to the transverse direction. There are two dispersion regimes: transverse dispersion ($t \gg \tau_{\rm T}$) and longitudinal dispersion ($t \gg \tau_{\rm L}$). In the regime of complete dispersion one obtains

$$\sigma_{\rm xL}(t) = \sigma_{\rm x}\frac{t}{\tau}\frac{1}{\gamma_{\mathbf{p}}^3}, \quad t \gg \tau_{\rm L}, \tag{20}$$

$$\sigma_{\rm xT}(t) = \sigma_{\rm x}\frac{t}{\tau}\frac{1}{\gamma_{\mathbf{p}}}, \quad t \gg \tau_{\rm T}. \tag{21}$$

It should come as no surprise, however, that the noncovariant wave packet in (12) might fail in predicting relativistic effects. Indeed, as we will demonstrate in Subsec. 1.2, a relativistically covariant version of the Gaussian wave packet predicts an alternate dependence on the longitudinal dispersion. Therefore, we will postpone a qualitative and quantitative discussion of dispersion effects until Subsec. 1.2.

Let us now see how these dispersions lead to a $1/|\mathbf{x}|^2$ suppression of the time-integrated flux density. For simplicity we will perform the calculations in the rest frame of the wave packet. The flux density reads

$$\mathbf{j}_{\rm G}(t,\mathbf{x}) = -i\left(\psi_{\rm G}^*(\mathbf{x},t)\nabla\psi_{\rm G}(\mathbf{x},t) - \psi_{\rm G}(\mathbf{x},t)\nabla\psi_{\rm G}^*(\mathbf{x},t)\right) = \\ = \frac{\mathbf{x}\,t/\tau \exp\left[-\dfrac{\mathbf{x}^2}{2\sigma_{\rm x}^2(1+t^2/\tau^2)}\right]}{(2\pi)^{3/2}2m\sigma_{\rm x}^5\left(1+t^2/\tau^2\right)^{5/2}}. \tag{22}$$

An explicit calculation of the time integral $\int\limits_0^\infty dt\,\mathbf{j}_{\rm G}(t,\mathbf{x})$ allows us to observe that, in the regime $|\mathbf{x}|^2 \gg \sigma_{\rm x}^2$, one has the following transparent formula:

$$\mathbf{\Phi}_{\rm G}(\mathbf{x}) = \int\limits_0^\infty dt\,\mathbf{j}_{\rm G}(t,\mathbf{x}) = \frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}. \tag{23}$$

The corrections to (23) are suppressed by $\mathrm{e}^{-\mathbf{x}^2/2\sigma_{\rm x}^2}$.

1.2. Covariant Gaussian Wave Packet. Let us note that relativistically covariant wave packet does not neccessarily imply near speed of light velocity of the latter. A relativistically covariant Gaussian wave packet was considered in [3, 4]. The proposed $\phi(\mathbf{k}) = \phi_{\rm RG}(\mathbf{k})$ function is explicitly Lorentz-invariant and reads as follows:

$$\phi_{\rm RG}(\mathbf{k}) = N_{\rm RG}\exp\left[\frac{(p-k)^2}{4\sigma_{\rm p}^2}\right]. \tag{24}$$

While an exact formula for the corresponding space–time wave function $\psi(x) = \psi_{\rm RG}(x)$ was obtained in [3, 4], we proceed here in an approximate fashion in order to more quickly attain the main results. Let us again exploit the smallness of $\sigma_{\rm p}^2$ and use (13) to produce

$$\phi_{\rm RG}(\mathbf{k}) = \sqrt{2m}\left(\frac{2\pi}{\sigma_{\rm p}^2}\right)^{3/4}\exp\left[-\frac{(\mathbf{p}-\mathbf{k}_{\rm L})^2}{4\sigma_{\rm pL}^2} - \frac{\mathbf{k}_{\rm T}^2}{4\sigma_{\rm pT}^2}\right], \tag{25}$$

where

$$\sigma_{\rm pL}^2 = \sigma_{\rm p}^2\gamma_{\mathbf{p}}^2, \quad \sigma_{\rm pT}^2 = \sigma_{\rm p}^2 \tag{26}$$

and the relativistically invariant normalization constant was obtained with the help of (7). The coordinate wave function $\psi_{\rm RG}(x)$ can be obtained in the same way as we proceeded for (14):

$$\psi_{\rm RG}(x) = \frac{\exp\left[-ipx - \dfrac{(\mathbf{x}_{\rm L}-\mathbf{v}t)^2}{4\sigma_{\rm xL}^2(1+it/\tau_{\mathbf{p}})} - \dfrac{\mathbf{x}_{\rm T}^2}{4\sigma_{\rm xT}^2(1+it/\tau_{\mathbf{p}})}\right]}{(2\pi)^{3/4}\sqrt{2m}\sigma_{\rm x}^3(1+it/\tau_{\mathbf{p}})^{3/2}}, \tag{27}$$

where

$$\sigma_{\rm xL}^2 = \frac{1}{4\sigma_{\rm pL}^2} = \frac{\sigma_{\rm x}^2}{\gamma_{\mathbf{p}}^2}, \quad \sigma_{\rm xT}^2 \quad = \frac{1}{4\sigma_{\rm pT}^2} = \sigma_{\rm x}^2, \quad \tau_{\mathbf{p}} = \tau\gamma_{\mathbf{p}} \tag{28}$$

with $\sigma_{\rm x}^2 = 1/4\sigma_{\mathbf{p}}^2$ and $\tau = 2m\sigma_{\rm x}^2$, just as in the case of the noncovariant wave packet. Similarly to Subsec. 1.1 one can obtain the longitudinal and transverse dispersions as functions of time

$$\sigma_{\rm xL}^2(t) = \sigma_{\rm xL}^2(1+t^2/\tau_{\mathbf{p}}^2), \tag{29}$$

$$\sigma_{\rm xT}^2(t) = \sigma_{\rm xT}^2(1+t^2/\tau_{\mathbf{p}}^2), \tag{30}$$

where $\sigma_{\rm xL}^2$ and $\sigma_{\rm xT}^2$ are given by (28). Now in the regime of complete dispersion ($t \gg \tau$) one has

$$\sigma_{\rm xL}(t) = \sigma_{\rm xL}(0)\frac{t}{\tau_{\mathbf{p}}}, \tag{31}$$

$$\sigma_{\rm xT}(t) = \sigma_{\rm xT}(0)\frac{t}{\tau_{\mathbf{p}}}, \tag{32}$$

$$\sigma_{\rm xL}(t) = \frac{1}{\gamma_{\mathbf{p}}}\sigma_{\rm xT}(t). \tag{33}$$

Comparing (31)–(33) to (20), (21) one might observe a problem with the noncovariant model's $\varphi_{\rm G}(\mathbf{k})$ (see (12)), which wrongly predicts, by the factor $1/\gamma_{\mathbf{p}}^2$, too slow a dispersion in the longitudinal direction as compared to the covariant model $\phi_{\rm RG}(\mathbf{k})$ (see (24)) where both the longitudinal and transverse dispersions have the same rate. However, since the size of the longitudinal spatial width is smaller than the corresponding transverse width by a factor of $\gamma_{\mathbf{p}}$, the absolute value of the former always remains smaller.

In order to give some quantitative estimates of both the longitudinal and transverse dispersions, let us consider three examples: an electron, a neutrino with a mass of 0.1 eV, and a «classical» particle with a mass of 1 g. Let us assume that initially these «particles» had in their rest frames $\sigma_{\rm x} = 1$ μm and see how long it will take to double the corresponding longitudinal and transverse sizes. We will consider two cases as an example: the particle is at rest, or has a full energy equal to 1 GeV. We summarize $\tau_{\rm L}$ and $\tau_{\rm T}$ in Table for noncovariant and τ for covariant Gaussian models (see (20), (21) and (31)–(33)).

$\tau_{\rm L}$ and $\tau_{\rm T}$ for particles with masses 0.5 MeV, 0.1 eV and a «classical» particle with a mass of 1 g. It is assumed that the particles are either at rest or have a total energy of 1 GeV. Estimates are given for both the noncovariant and covariant Gaussian models

Mass	$\gamma_{\mathbf{p}}$	Noncovariant		Covariant
		$\tau_{\rm L}$	$\tau_{\rm T}$	$\tau_{\mathbf{p}}$
$0.5 \cdot 10^6$ eV	1	$5 \cdot 10^{-8}$ s	$5 \cdot 10^{-8}$ s	$5 \cdot 10^{-8}$ s
$0.5 \cdot 10^6$ eV	$2 \cdot 10^3$	$4 \cdot 10^2$ s	10^{-4} s	10^{-4} s
0.1 eV	1	10^{-14} s	10^{-14} s	10^{-14} s
0.1 eV	10^{10}	10^{16} s	10^{-4} s	10^{-4} s
1 g	1	$3 \cdot 10^{11}$ y	$3 \cdot 10^{11}$ y	$3 \cdot 10^{11}$ y

As one can see from this table, the coordinate wave functions $\psi(x)$ of particles with microscopic masses quickly disperse in the rest frames of the particles. However, the predictions of the noncovariant and covariant Gaussian models are sharply different for the longitudinal dispersion rates. Essentially, the noncovariant model produces too slow a longitudinal dispersion (by a factor $\gamma_{\mathbf{p}}^2$) compared to the covariant model. This makes a dramatic difference. For example, a particle with neutrino mass on the order of 0.1 eV does not disperse longitudinally during the lifetime of the Universe according to the noncovariant model, while it disperses quite quickly (during 10^{-4} s) according to the covariant model.

Let us note also that the coordinate wave functions $\psi(x)$ of a «particle» with a mass on the order of 1 g, initially bound within a space of 1 μm, disperse within times significantly exceeding the lifetime of the Universe, thus bridging quantum and classical physics.

It is worth noting that the relativistic wave-packet model (24) also displays similar behavior at $\mathbf{x}^2 \gg \sigma_{\rm x}^2$ as in (23). As we will show in the next section this is a general property of wave packets of an arbitrary form.

2. WAVE PACKET OF AN ARBITRARY FORM

In this section it will be beneficial to use the following representation of $j_\mu(x)$, which can be obtained with the help of (4):

$$\rho(x) = \int d\mathbf{k}\, d\mathbf{q}\, \Pi(\mathbf{k}, \mathbf{q})\, {\rm e}^{-i(k-q)x}, \tag{34}$$

$$\mathbf{j}(x) = \int d\mathbf{k}\, d\mathbf{q}\, \mathbf{J}(\mathbf{k}, \mathbf{q})\, {\rm e}^{-i(k-q)x}, \tag{35}$$

where

$$\Pi(\mathbf{k},\mathbf{q}) = \frac{\phi(\mathbf{k})\,\phi(\mathbf{q})}{(2\pi)^6 2E_{\mathbf{k}}2E_{\mathbf{q}}}\left[E_{\mathbf{k}}+E_{\mathbf{q}}\right], \tag{36}$$

$$\mathbf{J}(\mathbf{k},\mathbf{q}) = \frac{\phi(\mathbf{k})\,\phi(\mathbf{q})}{(2\pi)^6 2E_{\mathbf{k}}2E_{\mathbf{q}}}\left[\mathbf{k}+\mathbf{q}\right]. \tag{37}$$

2.1. Trajectory and Dispersion of the Wave Packet. It is well known that the mean coordinate of the wave packet follows the classical trajectory. Indeed, explicit calculation for an arbitrary form of real-valued $\phi(\mathbf{k})$ yields

$$\langle\mathbf{x}\rangle = \int d\mathbf{x}\,\rho(t,\mathbf{x})\,\mathbf{x} = \int \frac{d\mathbf{k}\,\phi^2(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}}\frac{\mathbf{k}}{E_{\mathbf{k}}}t = \langle\mathbf{v}\rangle t, \tag{38}$$

where $\langle\mathbf{v}\rangle$ is given by (8).

Let us now examine the time dependence of the coordinate dispersion. By definition, the square of the spatial dispersion reads

$$\sigma_{\mathrm{x}}^2(t) = \langle\mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2. \tag{39}$$

Performing calculations for $\langle\mathbf{x}^2\rangle$ yields

$$\langle\mathbf{x}^2\rangle = \int d\mathbf{x}\,\mathbf{x}^2\rho(t,\mathbf{x}) = t^2\int\frac{d\mathbf{k}\,\phi^2(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}}\mathbf{v}_{\mathbf{k}}^2 - \int\frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}}\frac{\partial^2\phi(\mathbf{k})}{\partial\mathbf{k}^2} + \\ + \int\frac{d\mathbf{k}}{(2\pi)^3}\left[\phi^2(\mathbf{k})\left(\frac{m^2}{4E_{\mathbf{k}}^5} - \frac{\mathbf{v}_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3}\right) + \frac{\mathbf{v}_{\mathbf{k}}}{4E_{\mathbf{k}}^2}\frac{\partial\phi^2(\mathbf{k})}{\partial\mathbf{k}}\right]. \tag{40}$$

Taking into account the integral

$$\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\mathbf{v}_{\mathbf{k}}}{4E_{\mathbf{k}}^2}\frac{\partial\phi^2(\mathbf{k})}{\partial\mathbf{k}} = -\int\frac{d\mathbf{k}\,\phi^2(\mathbf{k})}{(2\pi)^3}\frac{\partial}{\partial\mathbf{k}}\frac{\mathbf{v}_{\mathbf{k}}}{4E_{\mathbf{k}}^2} = -\int\frac{d\mathbf{k}\,\phi^2(\mathbf{k})}{(2\pi)^3}\left(\frac{m^2}{4E_{\mathbf{k}}^5} - \frac{\mathbf{v}_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3}\right), \tag{41}$$

the last line of (40) is precisely cancelled. The integral

$$-\int\frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}}\frac{\partial^2\phi(\mathbf{k})}{\partial\mathbf{k}^2}$$

in (40) does not depend on space–time coordinates. It has the dimensions of the 3-coordinate squared and it is not invariant under the Lorentz transformations. Let us denote it by

$$\sigma_{\mathrm{x}}^2 = -\int\frac{d\mathbf{k}\,\phi(\mathbf{k})}{(2\pi)^3 2E_{\mathbf{k}}}\frac{\partial^2\phi(\mathbf{k})}{\partial\mathbf{k}^2}. \tag{42}$$

Therefore,

$$\langle\mathbf{x}^2\rangle = \sigma_{\mathrm{x}}^2 + \langle\mathbf{v}^2\rangle t^2 \tag{43}$$

where

$$\langle\mathbf{v}^2\rangle = \int\frac{d\mathbf{k}\,|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}}\mathbf{v}_{\mathbf{k}}^2. \tag{44}$$

Thus, the square of the dispersion of the coordinates reads

$$\begin{aligned}\sigma_{\mathrm{x}}^2(t) &= \sigma_{\mathrm{x}}^2 + \left(\langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2\right) t^2 \\ &= \sigma_{\mathrm{x}}^2 + \sigma_{\mathbf{v}}^2 t^2,\end{aligned} \tag{45}$$

where $\sigma_{\mathbf{v}}^2 = \langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2$ is the velocity dispersion. The general result (45) being one of the main results of this paper is not however new. It was already obtained in [5] where the derivation was unfortunately somewhat inconsistent as the authors asummed $|\psi(t, \mathbf{x})|^2$ to be a probability density function, while $\rho(t, \mathbf{x})$ defined in (6) instead should be used for the scalar relativistic particle.

We will now examine how the coordinate dispersion depends on relativistic effects, keeping in mind the discussions in Subsecs. 1.1 and 1.2 concerning different predictions for the longitudinal dispersion of the noncovariant and covariant Gaussian models of wave packets.

First, we rewrite $\sigma_{\mathbf{v}}^2$ as follows:

$$\begin{aligned}\sigma_{\mathbf{v}}^2 = \langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2 &= \langle \mathbf{v}_{\mathrm{T}}^2 \rangle + \langle \mathbf{v}_{\mathrm{L}}^2 \rangle - \langle \mathbf{v} \rangle^2 \\ &= \langle \mathbf{v}_{\mathrm{T}}^2 \rangle + \langle (\mathbf{v}_{\mathrm{L}} - \langle \mathbf{v} \rangle)^2 \rangle,\end{aligned} \tag{46}$$

where $\langle \mathbf{v}_{\mathrm{T}}^2 \rangle$ and $\langle \mathbf{v}_{\mathrm{L}}^2 \rangle$ are, respectively, the means of the squares of the longitudinal and transverse projections of wave-packet velocities relative to the mean velocity vector. Rewriting $\langle \mathbf{v}_{\mathrm{T}}^2 \rangle$ and $\langle (\mathbf{v}_{\mathrm{L}} - \langle \mathbf{v} \rangle)^2 \rangle$ using the variables in the rest frame of the wave packet

$$E_{\mathbf{k}} = \gamma_{\langle \mathbf{v} \rangle} (E_{\mathbf{k}}^* + \langle \mathbf{v} \rangle \mathbf{k}_{\mathrm{L}}^*), \tag{47}$$

$$\mathbf{k}_{\mathrm{L}} = \gamma_{\langle \mathbf{v} \rangle} (\mathbf{k}_{\mathrm{L}}^* + \langle \mathbf{v} \rangle E_{\mathbf{k}^*}), \quad \mathbf{k}_{\mathrm{T}} = \mathbf{k}_{\mathrm{T}}^*, \tag{48}$$

$$\mathbf{v}_{\mathbf{k}\mathrm{L}} = \frac{\mathbf{v}_{\mathbf{k}\mathrm{L}}^* + \langle \mathbf{v} \rangle}{1 + \mathbf{v}_{\mathbf{k}\mathrm{L}}^* \langle \mathbf{v} \rangle}, \quad \gamma_{\langle \mathbf{v} \rangle} = \frac{1}{\sqrt{1 - \langle \mathbf{v} \rangle^2}} \tag{49}$$

as follows:

$$\langle \mathbf{v}_{\mathrm{T}}^2 \rangle = \int \frac{d\mathbf{k}\, |\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}} \mathbf{v}_{\mathbf{k}\mathrm{T}}^2 = \frac{1}{\gamma_{\langle \mathbf{v} \rangle}^2} \int \frac{d\mathbf{k}^*\, |\phi(\mathbf{k}^*)|^2}{(2\pi)^3 2E_{\mathbf{k}^*}} \frac{\mathbf{v}_{\mathbf{k}\mathrm{T}}^{*2}}{(1 + \langle \mathbf{v} \rangle \mathbf{v}^*{}_{\mathbf{k}\mathrm{L}})^2} \tag{50}$$

and

$$\langle (\mathbf{v}_{\mathrm{L}} - \langle \mathbf{v} \rangle)^2 \rangle = \int \frac{d\mathbf{k}\, |\phi(\mathbf{k})|^2}{(2\pi)^3 2E_{\mathbf{k}}} (\mathbf{v}_{\mathbf{k}\mathrm{L}} - \langle \mathbf{v} \rangle)^2 = \frac{1}{\gamma_{\langle \mathbf{v} \rangle}^4} \int \frac{d\mathbf{k}^*\, |\phi(\mathbf{k}^*)|^2}{(2\pi)^3 2E_{\mathbf{k}^*}} \frac{\mathbf{v}_{\mathbf{k}\mathrm{L}}^{*2}}{(1 + \langle \mathbf{v} \rangle \mathbf{v}^*{}_{\mathbf{k}\mathrm{L}})^2}. \tag{51}$$

Comparing (50) to (51), one can observe that for narrow wave packets one gets, to the first order:

$$\langle \mathbf{v}_{\mathrm{T}}^2 \rangle = \frac{1}{\gamma_{\langle \mathbf{v} \rangle}^2} \langle \mathbf{v}_{\mathrm{T}}^{*2} \rangle, \tag{52}$$

$$\langle (\mathbf{v}_{\mathrm{L}} - \langle \mathbf{v} \rangle)^2 \rangle = \frac{1}{\gamma_{\langle \mathbf{v} \rangle}^4} \langle \mathbf{v}_{\mathrm{L}}^{*2} \rangle. \tag{53}$$

Therefore, in the regime of complete dispersion, keeping in mind that in the rest frame of the wave packet $\langle \mathbf{v}_{\rm L}^{*2}\rangle = \langle \mathbf{v}_{\rm T}^{*2}\rangle/2 = (1/3)\langle \mathbf{v}^{*2}\rangle$ and using (45), one obtains

$$\sigma_{\rm xL}^2(t) = \frac{1}{3\gamma_{\langle \mathbf{v}\rangle}^4}\langle \mathbf{v}^{*2}\rangle t^2, \tag{54}$$

$$\sigma_{\rm xT}^2(t) = \frac{2}{3\gamma_{\langle \mathbf{v}\rangle}^2}\langle \mathbf{v}^{*2}\rangle t^2, \tag{55}$$

$$\sigma_{\rm xL}^2(t) = \frac{1}{2\gamma_{\langle \mathbf{v}\rangle}^2}\sigma_{\rm xT}^2(t) \tag{56}$$

in agreement with calculations performed for the covariant model (see (31)–(33)). The seemingly extra factor of $1/2$ in (56) is due to the cumulative nature of the definition in (39), which adds together all projections of the dispersion. Apparently, in the rest frame of the wave packet

$$\sigma_{\rm x}^2(t) = \sigma_{\rm xL}^2(t) + \sigma_{\rm xT}^2(t) = \langle \mathbf{v}^{*2}\rangle t^2.$$

Let us examine the following question: if the Gaussian wave packet can be shown to exhibit an asymptotic behavior of $\mathbf{\Phi}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|^3$ (see (23)), does this also hold true in the general case? As we will see in Subsec. 2.2 this is indeed true. Moreover, keeping in mind that probability $\rho(x)$ and flux $\mathbf{j}(x)$ densities are closely related to each other, it implies that a similar asymptotic behavior should also apply to the integral $\int\limits_0^\infty dt\rho(t,\mathbf{x})$. Indeed, for the wave function with definite 4-momentum ($\psi(x) = N\,{\rm e}^{-ipx}$), the relation is obvious:

$$\rho(x) = 2E_{\mathbf{p}}|N|^2, \quad \mathbf{j}(x) = 2\mathbf{p}|N|^2, \quad \mathbf{j}(x) = \mathbf{v}\rho(x). \tag{57}$$

Apparently, for narrow wave packets, a relation similar to (57) should be valid as well. In Subsec. 2.3 we will explicitly calculate the asymptotic behavior of the time-integrated probability densities.

2.2. Asymptotic Behavior of the Time-Integrated Flux Density. Let us compute here

$$\mathbf{\Phi}(\mathbf{x}) \equiv \int\limits_0^\infty dt\,\mathbf{j}(t,\mathbf{x}). \tag{58}$$

In order to integrate over the time in (58), let us use the following formula:

$$\int\limits_0^\infty dt\,{\rm e}^{\pm i\alpha t} = \pi\left(\delta(\alpha) \pm \frac{i}{\pi}\mathcal{P}\frac{1}{\alpha}\right), \tag{59}$$

where $\mathcal{P}$ represents the Cauchy principal value of the integral.

Therefore,

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}_1(\mathbf{x}) + \mathbf{\Phi}_2(\mathbf{x}), \tag{60}$$

where

$$\mathbf{\Phi}_1(\mathbf{x}) = \pi\int d\mathbf{k}\,d\mathbf{q}\,\mathbf{J}(\mathbf{k},\mathbf{q})\,\delta(E_{\mathbf{k}} - E_{\mathbf{q}})\cos\left[(\mathbf{k}-\mathbf{q})\mathbf{x}\right], \tag{61}$$

$$\mathbf{\Phi}_2(\mathbf{x}) = \int d\mathbf{k}\,d\mathbf{q}\,\mathbf{J}(\mathbf{k},\mathbf{q})\,\mathcal{P}\left(\frac{1}{E_{\mathbf{k}} - E_{\mathbf{q}}}\right)\sin\left[(\mathbf{k}-\mathbf{q})\mathbf{x}\right]. \tag{62}$$

It is easy to see that in the rest frame of the wave packet $\mathbf{\Phi}_1(\mathbf{x}) = 0$, because changing the integration variables $\mathbf{k} \to -\mathbf{k}$ and $\mathbf{q} \to -\mathbf{q}$ changes the sign of the integrand $\mathbf{J}(\mathbf{k},\mathbf{q}) \to \mathbf{J}(-\mathbf{k},-\mathbf{q}) = -\mathbf{J}(\mathbf{k},\mathbf{q})$, and in the rest frame of the wave packet $\phi(\mathbf{k})$ depends only on the absolute value of $\mathbf{k}$ and not on its direction.

For the remaining integrals in (62) one could work out the angular integrations in

$$\mathbf{\Phi}_2(\mathbf{x}) = \mathcal{P}\int_0^\infty d\mathrm{k}\, d\mathrm{q} \int d\mathbf{n}_1\, d\mathbf{n}_2 \frac{\mathrm{k}^2\mathrm{q}^2\phi(\mathrm{k}\mathbf{n}_1)\,\phi(\mathrm{q}\mathbf{n}_2)}{(2\pi)^6 2E_\mathrm{k} 2E_\mathrm{q}} \times \\ \times \left[\mathrm{k}\mathbf{n}_1 + \mathrm{q}\mathbf{n}_2\right] \frac{1}{E_\mathrm{k} - E_\mathrm{q}} \sin\left[\left(\mathrm{k}\mathbf{n}_1 - \mathrm{q}\mathbf{n}_2\right)\mathbf{x}\right] \quad (63)$$

by again keeping in mind that $\phi(\mathrm{k}\mathbf{n})$ does not depend on the direction vector $\mathbf{n}$ in the rest frame of the wave packet and using the following:

$$\int d\mathbf{n}_1 d\mathbf{n}_2 \left[\mathrm{k}\mathbf{n}_1 + \mathrm{q}\mathbf{n}_2\right] \sin\left[\left(\mathrm{k}\mathbf{n}_1 - \mathrm{q}\mathbf{n}_2\right)\mathbf{x}\right] = \\ = -\frac{(4\pi)^2}{2\mathrm{kq}} \frac{\mathbf{x}}{|\mathbf{x}|^3} \left[(\mathrm{k}-\mathrm{q})\sin\left((\mathrm{k}+\mathrm{q})|\mathbf{x}|\right) - (\mathrm{k}+\mathrm{q})\sin\left((\mathrm{k}-\mathrm{q})|\mathbf{x}|\right)\right]. \quad (64)$$

Using (64) and an obvious identity

$$\frac{1}{E_\mathrm{k} - E_\mathrm{q}} = \frac{E_\mathrm{k} + E_\mathrm{q}}{(\mathrm{k}-\mathrm{q})(\mathrm{k}+\mathrm{q})},$$

(63) can be written in the following way:

$$\mathbf{\Phi}_2(\mathbf{x}) = -\frac{1}{32\pi^4} \frac{\mathbf{x}}{|\mathbf{x}|^3} \mathcal{P}\int_0^\infty d\mathrm{k}\, d\mathrm{q} \frac{\mathrm{kq}(E_\mathrm{k} + E_\mathrm{q})\phi(\mathrm{k})\phi(\mathrm{q})}{E_\mathrm{k} E_\mathrm{q}} \times \\ \times \left[\frac{\sin\left((\mathrm{k}+\mathrm{q})|\mathbf{x}|\right)}{\mathrm{k}+\mathrm{q}} - \frac{\sin\left((\mathrm{k}-\mathrm{q})|\mathbf{x}|\right)}{\mathrm{k}-\mathrm{q}}\right]. \quad (65)$$

One might notice that the singularities introduced by a generalized function $\mathcal{P}(E_\mathbf{k} - E_\mathbf{q})^{-1}$ due to (59) disappear thanks to the corresponding $\sin\left((\mathrm{k} \pm \mathrm{q})|\mathbf{x}|\right)$ in the numerator of the integrals in (65). While the remaining integrals in (65) can be calculated only for explicit forms of the function $\phi(\mathrm{k})$, we could proceed further by noting that, in the limit $|\mathbf{x}| \to \infty$, the terms

$$\frac{\sin\left((\mathrm{k} \pm \mathrm{q})|\mathbf{x}|\right)}{\mathrm{k} \pm \mathrm{q}}$$

could be replaced by delta functions with the argument $\mathrm{k} \pm \mathrm{q}$ due to the following relation:

$$\lim_{x\to\infty} \frac{\sin\,\alpha x}{\alpha} = \pi\delta(\alpha). \quad (66)$$

The first delta function $\delta(\mathrm{k}+\mathrm{q})$ cancels the integral because the integrand is zero at $\mathrm{q} = \mathrm{k} = 0$. A non-zero contribution comes from the second delta function $\delta(\mathrm{k} - \mathrm{q})$. Therefore,

$$\mathbf{\Phi}_2(\mathbf{x}) = \frac{\mathbf{x}}{16\pi^3|\mathbf{x}|^3} \int_0^\infty d\mathrm{k} \frac{\mathrm{k}^2\phi^2(\mathrm{k})}{E_\mathrm{k}} = \frac{1}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad (67)$$

where one might notice that the remaining integral in (67) reduces to the normalization integral in (7) (multiplied by $4\pi^2$) in the rest frame of the wave packet.

To obtain (67) we used the limit $|\mathbf{x}| \to \infty$. At what distance $|\mathbf{x}|$ will (67) be a good approximation? A dimensional analysis suggests that if $\phi(\mathrm{k})$ can be characterized by a certain momentum «width» σ_p, then this approximation works with a good accuracy for $|\mathbf{x}| \gg 1/\sigma_\mathrm{p}$. Therefore, we have just proved that, for a wave packet of an arbitrary form, the time-integrated flux density displays the asymptotic behavior:

$$\mathbf{\Phi}(\mathbf{x}) = \frac{1}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3} \text{ at } |\sigma_\mathrm{p}\mathbf{x}| \gg 1. \tag{68}$$

2.3. Asymptotic Behavior of the Time-Integrated Probability Density. We will now inspect the asymptotic nature of the time-integrated probability density

$$P(\mathbf{x}) \equiv \int\limits_0^\infty dt\, \rho(\mathbf{t},\mathbf{x}) = P_1(\mathbf{x}) + P_2(\mathbf{x}), \tag{69}$$

where $P_{1,2}(\mathbf{x})$ are defined via (59) as follows:

$$P_1(\mathbf{x}) = \int d\mathbf{k}\, d\mathbf{q}\, \Pi(\mathbf{k},\mathbf{q})\, \pi\delta(E_\mathbf{k} - E_\mathbf{q}) \cos\left[(\mathbf{k}-\mathbf{q})\mathbf{x}\right], \tag{70}$$

$$P_2(\mathbf{x}) = \int d\mathbf{k}\, d\mathbf{q}\, \Pi(\mathbf{k},\mathbf{q})\, \mathcal{P}\frac{1}{E_\mathbf{k} - E_\mathbf{q}} \sin\left[(\mathbf{k}-\mathbf{q})\mathbf{x}\right]. \tag{71}$$

In the rest frame of the wave packet $P_2(\mathbf{x}) = 0$. As is easy to see, changing the variables $\mathbf{k} \to -\mathbf{k}$ and $\mathbf{q} \to -\mathbf{q}$ changes the sign of the integrand, while the integration limits remain the same. Let us compute $P_1(\mathbf{x})$:

$$P_1(\mathbf{x}) = \frac{1}{|\mathbf{x}|^2}\int\limits_0^\infty \frac{d\mathrm{k}\,\mathrm{k}\phi^2(\mathrm{k})}{(2\pi)^3}\sin^2(\mathrm{k}|\mathbf{x}|) = \frac{1}{4\pi|\mathbf{x}|^2}\int \frac{d\mathbf{k}\,\phi^2(\mathbf{k})}{(2\pi)^3 2E_\mathbf{k}}\frac{E_\mathbf{k}}{\mathrm{k}} - \delta P_1(\mathbf{x}), \tag{72}$$

where

$$\delta P_1(\mathbf{x}) = \frac{1}{|\mathbf{x}|^2}\frac{\partial}{\partial|\mathbf{x}|}\int\limits_0^\infty \frac{d\mathrm{k}\,\mathrm{k}\phi^2(\mathrm{k})}{(2\pi)^3}\frac{\sin(2\mathrm{k}|\mathbf{x}|)}{2\mathrm{k}}. \tag{73}$$

One may note that

$$\lim_{|\mathbf{x}|\to\infty} \delta P_1(\mathbf{x}) = 0$$

because in this limit

$$\lim_{|\mathbf{x}|\to\infty} \frac{\sin\, 2\mathrm{k}|\mathbf{x}|}{2\mathrm{k}} = \pi\delta(2\mathrm{k})$$

and

$$\lim_{|\mathbf{x}|\to\infty}\int\limits_0^\infty \frac{d\mathrm{k}\,\mathrm{k}\phi^2(\mathrm{k})}{(2\pi)^3}\frac{\sin(2\mathrm{k}|\mathbf{x}|)}{2\mathrm{k}} = \left.\frac{\mathrm{k}\phi^2(\mathrm{k})}{2(2\pi)^3}\right|_{\mathrm{k}=0} = 0.$$

Therefore,

$$P(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|^2}\left\langle|\mathbf{v}|^{-1}\right\rangle \text{ at } |\sigma_\mathrm{p}\mathbf{x}| \gg 1. \tag{74}$$

Thus, we have shown that the time-integrated probability density also scales as $1/|\mathbf{x}|^2$ with a coefficient of proportionality equal to the mean of the inverse absolute value of the wave-packet velocity, as was suggested by (57):

$$\int\limits_0^\infty dt\,\rho(t,\mathbf{x}) = \left\langle|\mathbf{v}|^{-1}\right\rangle \int\limits_0^\infty dt\,|\mathbf{j}(t,\mathbf{x})|. \tag{75}$$

Let us note, however, that despite the fact that the mean velocity in the rest frame of the wave packet is zero, its mean absolute value (and therefore its inverse) is not zero. This is in contrast to the plain wave, where the velocity in the rest frame of a particle is zero, making it impossible for (75) to be the plain wave solution.

As we have shown in this section, the asymptotic behavior of $1/|\mathbf{x}|^2$ is valid for both the time-integrated probability density $\rho(t,\mathbf{x})$ and the flux density $\mathbf{j}(t,\mathbf{x})$ for wave packets of an arbitrary form satisfying the Klein–Gordon equation. We will now show that the $1/|\mathbf{x}|^2$ behavior applies to more general situations following from the continuity equation, which also holds true for solutions to both the Schroedinger and Dirac equations.

3. THE CONTINUITY EQUATION AND $1/|\mathbf{x}|^2$

The continuity equation for a quantum state with $\rho(t,\mathbf{x}')$, being the probability density to observe the particle in point $\mathbf{x}'$ at time t, and $\mathbf{j}(t,\mathbf{x}')$, the corresponding flux density, is well known:

$$\frac{\partial\rho(t,\mathbf{x}')}{\partial t} + \boldsymbol{\nabla}\mathbf{j}(t,\mathbf{x}') = 0. \tag{76}$$

Equation (76) could be rewritten as follows:

$$\frac{\partial}{\partial t}\int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(t,\mathbf{x}') = -\int\limits_S d\mathbf{S}\,\mathbf{j}(t,\mathbf{x}'), \tag{77}$$

where the integration is limited by a sphere S of radius $|\mathbf{x}|$. If the radius $|\mathbf{x}|$ is sufficiently large and time t is so small that the wave packet is not dispersed significantly (both conditions are satisfied if $|\mathbf{x}| \gg \sigma_\mathrm{x}(t)$), then one might expect that the integral is almost saturated:

$$\int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(t,\mathbf{x}') \approx 1$$

and

$$\frac{\partial}{\partial t}\int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(t,\mathbf{x}')$$

vanishes. Therefore, the right-hand side of (77) also vanishes. This implies that the flux density should decrease faster than $1/|\mathbf{x}|^2$. The flux density example for the noncovariant Gaussian wave packet in (22) agrees with this conclusion.

Let us perform the time integration for the left- and right-hand sides between zero and infinity. Therefore,

$$\int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(\infty,\mathbf{x}') - \int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(0,\mathbf{x}') = -\int\limits_{S} d\mathbf{S}\,\mathbf{\Phi}(\mathbf{x}'), \tag{78}$$

where $\mathbf{\Phi}(\mathbf{x}') = \int\limits_0^\infty dt\,\mathbf{j}(t,\mathbf{x}')$ is the flux density integrated over the time. By definition

$$P(t,|\mathbf{x}|) \equiv \int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(t,\mathbf{x}') \tag{79}$$

gives the probability to find a particle within a sphere of radius $|\mathbf{x}|$ at time t. Apparently, due to dispersion of the wave packet, the probability to find a particle within any volume of a finite size tends to zero at $t\to\infty$ because the particle leaves the volume. On the other hand, at $|\mathbf{x}|$ much larger than the wave packet's «size» the value of $P(0,|\mathbf{x}|)$ is very close to unity because initially the wave packet is almost fully contained within a large enough volume. With these conditions (78) becomes

$$\int\limits_{|\mathbf{x}'|\leqslant|\mathbf{x}|} d\mathbf{x}'\rho(0,\mathbf{x}') = \int\limits_{S} d\mathbf{S}\,\mathbf{\Phi}(\mathbf{x}') \approx 1, \tag{80}$$

from which it follows immediately that in the rest frame of the wave packet

$$|\mathbf{\Phi}(|\mathbf{x}|)| = \frac{1}{4\pi|\mathbf{x}|^2}.$$

Therefore, the $1/|\mathbf{x}|^2$ dependence holds true for any wave-packet solution of the Klein–Gordon and/or Dirac equations with finite normalization (whereas plain waves do not have a finite norm). At any given moment in time the flux vanishes faster than $1/|\mathbf{x}|^2$.

4. DISCUSSIONS AND CONCLUSIONS

Let us briefly discuss the main points raised in this paper. A wave packet possesses some characteristics of a «particle» or a solid body, which has well-defined mean energy and momentum, while the wave packet is off-shell. In other words, its energy in the rest frame of the wave packet is not equal to the mass of the waves composing the packet. This can be easily seen from (11) in the rest frame:

$$\langle \mathrm{E}\rangle = \int \frac{d\mathbf{k}\,|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_\mathbf{k}} E_\mathbf{k} \geqslant m \int \frac{d\mathbf{k}\,|\phi(\mathbf{k})|^2}{(2\pi)^3 2E_\mathbf{k}} = m,$$

where we used the normalization (7).

A wave packet follows the classical trajectory for its mean 3-coordinate: $\langle \mathbf{x} \rangle = \langle \mathbf{v} \rangle t$. On the other hand, a particle described by the wave packet can be detected at any place in the Universe, though the probabilities near the classical trajectory and far away from it differ sharply.

Wave packets, by their nature, disperse over time. The dispersion is very different between the direction along the mean velocity and the transverse direction. We addressed this issue with the help of noncovariant and covariant Gaussian models, and found that the noncovariant wave packet predicts too slow a longitudinal dispersion relative to the covariant model. This makes a dramatic difference for ultrarelativistic particles like a neutrino with γ^{10}, which will not disperse significantly in the longitudinal direction during the lifetime of the Universe according to the noncovariant model. However, using the covariant model, and assuming an initial wave packet spatial size on the order of 1 μm, such a neutrino will disperse in about 10^{-4} s.

By performing calculations for a wave packet of an arbitrary form we confirmed in Sec. 2 that the relationship between the longitudinal and transverse dispersion times given by the covariant Gaussian model is correct. Moreover, we found a general formula for the dispersion of a wave packet of an arbitrary form and found that it linearly increases with time with a coefficient of proportionality equal to the wave packet's velocity dispersion.

Finally, we addressed the question of interpretation of the dispersion of a wave packet, which is often considered as a disadvantage in attempts to describe a «stable particle» whose wave function vanishes with time. Proceeding from simple examples of Gaussian wave packets describing a spinless particle, generalizing it to wave packets of an arbitrary form, and finally considering the continuity equation, we find that the time-integrated flux or probability density always displays an asymptotic behavior which is proportional to $1/|\mathbf{x}|^2$ in the rest frame of the wave packet, as one would expect for an ensemble of classical particles if its number density is normalized to the number of particles in the ensemble. The analysis of correspondence between an ensemble of one-particle wave packets and uniform flux of particles was also discussed in [2, 6].

As we have demonstrated in this paper, the origin of $1/|\mathbf{x}|^2$ law for quantum objects is their dispersion with time. For wave packets sufficiently narrow in time this law appears automatically provided the detection time is much longer than the wave-packet time width. On the other hand, accelerator experiments with beams very narrow in time or experiments at short enough distances might probe deviations from $1/|\mathbf{x}|^2$ law.

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