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THE EULER BOUND STATES: 8D QUANTUM OSCILLATOR

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The problem of eight-dimensional oscillator in the Euler coordinates is analyzed. The spherical and cylindrical bases are constructed, two representations for the coefficients of spherical-cylindrical and cylindrical-spherical interbasis expansion are proved, and the three-term recurrence relations generating a spheroidal basis for the eight-dimensional oscillator are established.

INTRODUCTION

It is known [1, 2] that the Hurwitz transformation [3] connects two fundamental problems of quantum mechanics: the eight-dimensional isotropic oscillator problem with the five-dimensional Coulomb problem.

The Hurwitz transformation can be written in the following form:

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2, \\ x_1 + ix_2 &= 2(u_0 - iu_1)(u_4 + iu_5) - 2(u_2 + iu_3)(u_6 - iu_7), \\ x_3 + ix_4 &= 2(u_0 - iu_1)(u_6 + iu_7) + 2(u_2 + iu_3)(u_4 - iu_5). \end{aligned} \tag{1}$$

Here $u_\mu \, (\mu = 0, 1, ..., 7)$ are the coordinates of the space $\mathbb{R}^8(\mathbf{u})$; and $x_i \, (i = 0, 1, ..., 4)$, of the space $\mathbb{R}^5(\mathbf{x})$. It is easily seen from (1) that the following equality holds:

$$u^4 = \left(u_0^2 + u_1^2 + \ldots + u_7^2\right)^2 = x_0^2 + x_1^2 + \ldots + x_4^2 = r^2, \tag{2}$$

which is called the Euler identity. According to [1], the connection of the Laplace operators in the spaces $\mathbb{R}^8$ and $\mathbb{R}^5$ has the form

$$\Delta_8 = 4r\Delta_5 - \frac{4}{r}\hat{J}^2, \tag{3}$$

where $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, and

$$\hat{J}_1 = \frac{i}{2}\left(u_1\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3} + \right.$$
$$\left. + u_5\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial u_5} + u_7\frac{\partial}{\partial u_6} - u_6\frac{\partial}{\partial u_7}\right),$$
$$\hat{J}_2 = \frac{i}{2}\left(u_2\frac{\partial}{\partial u_0} - u_3\frac{\partial}{\partial u_1} - u_0\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_3} - \right.$$
$$\left. - u_6\frac{\partial}{\partial u_4} + u_7\frac{\partial}{\partial u_5} + u_4\frac{\partial}{\partial u_6} - u_5\frac{\partial}{\partial u_7}\right),$$
$$\hat{J}_3 = \frac{i}{2}\left(u_3\frac{\partial}{\partial u_0} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_0\frac{\partial}{\partial u_3} - \right.$$
$$\left. - u_7\frac{\partial}{\partial u_4} - u_6\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_6} + u_4\frac{\partial}{\partial u_7}\right).$$

Using the explicit form of the operators one can prove by a direct calculation that the operators $\hat{J}_1$, $\hat{J}_2$, and $\hat{J}_3$ satisfy the commutation relations

$$\left[\hat{J}_a, \hat{J}_b\right] = i\epsilon_{abc}\hat{J}_c,$$

where a, b, and c are equal to 1, 2, and 3, respectively.

Now connect the eight-dimensional problem of isotropic oscillator

$$\left(-\frac{\hbar^2}{2\mu_0}\frac{\partial^2}{\partial u_\mu^2} + \frac{\mu_0\omega^2 u^2}{2}\right)\psi(\mathbf{u}) = E\psi(\mathbf{u}), \tag{4}$$

$$E = \hbar\omega\left(N+4\right), \qquad N = 0, 1, 2, ..., \tag{5}$$

where N is the principal quantum number, with the five-dimensional Coulomb problem. Substituting (3) into (4) and assuming that

$$\hat{J}_a\psi(\mathbf{u}) = 0,$$

we arrive at the equation for the five-dimensional Coulomb problem

$$\left(-\frac{\hbar^2}{2\mu_0}\frac{\partial^2}{\partial x_j^2} - \frac{e^2}{r}\right)\psi(\mathbf{x}) = \varepsilon\psi(\mathbf{x}), \tag{6}$$

where $\varepsilon = -\mu_0\omega^2/8$ and $4e^2 = E$. Moreover, it follows from (1) that $\psi(\mathbf{x})$ is the even function of variables u

$$\psi\left(\mathbf{x}(-\mathbf{u})\right) = \psi\left(\mathbf{x}(\mathbf{u})\right).$$

Therefore, any solution of (6), $\psi(\mathbf{x})$ can be expanded over a complete system of even solutions $\psi_{N\alpha}(\mathbf{u})$ (α is the remaining quantum numbers) of equation (4), i. e.,

$$\psi_n(\mathbf{x}) = \sum_{\alpha} C_{n\alpha}\psi_N(\mathbf{u}),$$

where

$$N = 2n. \tag{7}$$

One can easily be convinced that n coincides with the principal quantum number of the five-dimensional Coulomb problem. Indeed, substituting the relation $E = 4e^2$ and (7) into (5), we get

$$\omega_n = \frac{2e^2}{\hbar(n+2)}. \tag{8}$$

Thus, in our case, the oscillator energy is fixed and frequency ω is quantized. Now substituting (8) into the condition $\varepsilon = -\mu_0\omega^2/8$ we arrive at the expression

$$\varepsilon_n = -\frac{\mu_0 e^4}{2\hbar^2(n+2)^2}, \tag{9}$$

which determines the energy spectrum of the five-dimensional Coulomb problem [4].

1. SPHERICAL BOUND STATES

Determine the Euler eight-dimensional spherical coordinates as follows:

$$\begin{aligned}
u_0 + iu_1 &= u\cos\frac{\theta}{2}\sin\frac{\beta_T}{2}\,\mathrm{e}^{-i((\alpha_T-\gamma_T)/2)},\\
u_2 + iu_3 &= u\cos\frac{\theta}{2}\cos\frac{\beta_T}{2}\,\mathrm{e}^{i((\alpha_T+\gamma_T)/2)},\\
u_4 + iu_5 &= u\sin\frac{\theta}{2}\sin\frac{\beta_K}{2}\,\mathrm{e}^{i((\alpha_K-\gamma_K)/2)},\\
u_6 + iu_7 &= u\sin\frac{\theta}{2}\cos\frac{\beta_K}{2}\,\mathrm{e}^{-i((\alpha_K+\gamma_K)/2)},
\end{aligned} \tag{10}$$

where $0 \le u < \infty$, $0 \le \theta \le \pi$. In these coordinates, the differential elements of length and volume, and Laplace operator have the form

$$\begin{aligned}
dl_8^2 &= du^2 + \frac{u^2}{4}\left(d\theta^2 + \cos^2\frac{\theta}{2}dl_T^2 + \sin^2\frac{\theta}{2}dl_K^2\right),\\
dV_8 &= u^7\sin^3\theta\, du\, d\theta\, d\Omega_T\, d\Omega_K,
\end{aligned}$$

$$\Delta_8 = \frac{1}{u^7}\frac{\partial}{\partial u}\left(u^7\frac{\partial}{\partial u}\right) + \frac{4}{u^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) - \\ - \frac{4}{u^2\cos^2\theta/2}\hat{T}^2 - \frac{4}{u^2\sin^2\theta/2}\hat{K}^2,$$

where

$$dl_a^2 = d\alpha_a{}^2 + d\beta_a{}^2 + d\gamma_a{}^2 + 2\cos\beta_a d\alpha_a d\gamma_a, \quad d\Omega_a = \frac{1}{8}\sin\beta_a d\beta_a d\alpha_a d\gamma_a,$$

$$\hat{T}^2 = -\left[\frac{\partial^2}{\partial\beta_T^2} + \cot\beta_T\frac{\partial}{\partial\beta_T} + \frac{1}{\sin^2\beta_T}\left(\frac{\partial^2}{\partial\alpha_T{}^2} - 2\cos\beta_T\frac{\partial^2}{\partial\alpha_T\partial\gamma_T} + \frac{\partial^2}{\partial\gamma_T{}^2}\right)\right]$$

and $a = T$, K, and the operator $\hat{K}^2$ can be derived from the operator $\hat{T}^2$ by the substitution of $(\alpha_T, \beta_T, \gamma_T)$ by $(\alpha_K, \beta_K, \gamma_K)$.

In the coordinates (10), to the scheme of separation of variables for the eight-dimensional oscillator

$$V = \frac{\mu_0\omega^2 u^2}{2}$$

there corresponds the following factorization

$$\Psi^{\text{sph}} = R(u)Z(\theta)D_{tt'}^T(\alpha_T, \beta_T, \gamma_K)D_{kk'}^K(\alpha_K, \beta_K, \gamma_K),$$

where $D_{mm'}^j$ D are the Wigner functions. Taking into account that

$$\begin{aligned}\hat{T}^2 D_{tt'}^T(\alpha_T, \beta_T, \gamma_K) &= T(T+1)D_{tt'}^T(\alpha_T, \beta_T, \gamma_K),\\ \hat{K}^2 D_{kk'}^K(\alpha_K, \beta_K, \gamma_K) &= K(K+1)D_{kk'}^K(\alpha_K, \beta_K, \gamma_K),\end{aligned}$$

we arrive at the following pair of differential equations

$$\left[\frac{1}{\sin^3\theta}\frac{d}{d\theta}\left(\sin^3\theta\frac{d}{d\theta}\right) - \frac{T(T+1)}{\cos^2\theta/2} - \frac{K(K+1)}{\sin^2\theta/2} + \lambda(\lambda+3)\right]Z(\theta) = 0, \quad (11)$$

$$\left[\frac{1}{u^7}\frac{d}{du}\left(u^7\frac{d}{du}\right) - \frac{4\lambda(\lambda+3)}{u^2} + \frac{2\mu_0 E}{\hbar^2} - a^4u^2\right]R(u) = 0, \quad (12)$$

where $a = (\mu_0\omega\hbar)^{1/2}$ and the $\lambda(\lambda+3)$ non-negative constant of separation is the eigenvalue of the operator

$$\hat{\Lambda}^2 = -\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\cos^2\theta/2}\hat{T}^2 + \frac{1}{\sin^2\theta/2}\hat{K}^2. \quad (13)$$

First, let us consider equation (11). Passing in it to a new variable $y = (1-\cos\theta)/2$ we look for a solution in the following form

$$Z(y) = y^K\,(1-y)^T\,W(y).$$

Substituting the last relation into (11) we arrive at the hypergeometric equation

$$y(1-y)\frac{d^2W}{dy^2} + [\gamma - (\alpha+\beta+1)y]\,\frac{dW}{dy} - \alpha\beta W = 0$$

with $\alpha = -\lambda + K + T$, $\beta = \lambda + K + T + 3$, $2K+2$. Thus, we find that

$$Z_{\lambda KT}(\theta) = (1-\cos\theta)^K\,(1+\cos\theta)^T\,\times$$
$$\times\,{}_2F_1\left(-\lambda+K+T, \lambda+K+T+3; 2K+2; \frac{1-\cos\theta}{2}\right).$$

This solution has a good behaviour at $\theta = \pi$ if the series ${}_2F_1$ is finite, i. e.,

$$-\lambda + K + T = -n_\theta = 0, -1, -2, \ldots$$

Now using the formula

$${}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-y}{2}\right) = \frac{n!\Gamma(a+1)}{\Gamma(n+a+1)}P_n^{(a,b)}(y),$$

where $P_n^{(a,b)}(y)$ are the Jacobi polynomials, and taking account of the integral

$$\int_{-1}^{1}(1-y)^a(1+y)^b P_n^{(a,b)}(y)P_{n'}^{(a,b)}(y)dy =$$
$$= \frac{2^{a+b+1}}{2n+a+b+1}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{n!\Gamma(n+a+b+1)}\delta_{nn'},$$

normalized by the condition

$$\frac{1}{16}\int_0^\pi Z_{\lambda KT}(\theta)Z_{\lambda' KT}(\theta)\sin^3\theta d\theta = \delta_{\lambda\lambda'},$$

one can write the function $Z_{\lambda KT}(\theta)$ in the form

$$Z_{\lambda KT}(\theta) = \sqrt{\frac{2(2\lambda+3)(\lambda-K-T)!(\lambda+K+T+2)!}{(\lambda+K-T+1)!(\lambda-K+T+1)!}}\,\times$$
$$\times\left(\sin\frac{\theta}{2}\right)^{2K}\left(\cos\frac{\theta}{2}\right)^{2T}P_{\lambda-K-T}^{(2K+1,2T+1)}(\cos\theta). \quad (14)$$

Now we return to the radial equation. Upon substituting into (12)

$$R(u) = u^{2\lambda}\,\mathrm{e}^{-a^2u^2/2}\,f(u)$$

for the function $f(u)$ we derive the equation for the degenerate hypergeometric function

$$z\frac{d^2f}{dz^2} + (\gamma - z)\frac{df}{dz} - \alpha f = 0, \tag{15}$$

where $z = a^2u^2$, $\alpha = \lambda + 2 - E/2\hbar\omega$, and $\gamma = 2\lambda + 4$, i. e., the function $R(u)$ has the form

$$R(u) = u^{2\lambda}\,\mathrm{e}^{-a^2u^2/2}\,F\left(\lambda + 2 - \frac{E}{2\hbar\omega}; 2\lambda + 4; a^2u^2\right).$$

This solution has a good behaviour as $u \to \infty$ if the degenerate hypergeometric function is finite, i. e.,

$$\lambda + 2 - \frac{E}{2\hbar\omega} = -n_u = 0, -1, -2, \ldots$$

Hence it follows that

$$E = \hbar\omega(N + 4), \tag{16}$$

where $N = 2(n_u + \lambda)$. The solution of the radial equation (12) normalized by the condition

$$\int\limits_0^\infty u^7\,R_{N\lambda}(u)\,R_{N'\lambda}(u)du = \delta_{NN'}$$

has the form

$$R_{N\lambda}(u) = a^4\sqrt{\frac{2(N/2+\lambda+3)!}{(N/2-\lambda)!}}\,\frac{(au)^{2\lambda}}{(2\lambda+3)!}\times \\ \times\,\mathrm{e}^{-a^2u^2/2}\,F\left(-\frac{N}{2} + \lambda; 2\lambda + 4; a^2u^2\right). \tag{17}$$

The total wave function can be written as

$$\Psi^{\mathrm{sph}} = \sqrt{\frac{(2K+1)(2T+1)}{4\pi^4}}\times \\ \times\,R_{N\lambda}(u)\,Z_{\lambda KT}(\theta)D^T_{tt'}(\alpha_T, \beta_T, \gamma_K)D^K_{kk'}(\alpha_K, \beta_K, \gamma_K). \tag{18}$$

It is normalized by the condition

$$\int |\Psi^{\mathrm{sph}}|^2 \, dV_8 \;=\; 1.$$

When calculating the total normalization factor we have used the formula [5]

$$\int D^{j_2 *}_{m_2 m_2'}(\alpha,\beta,\gamma)\, D^{j_1}_{m_1 m_1'}(\alpha,\beta,\gamma)\, d\Omega = \frac{16\pi^2}{2j_1+1}\delta_{j_1 j_2}\,\delta_{m_1 m_2}\,\delta_{m'_1 m'_2}.$$

Thus, it can be stated that the spherical wave functions (18) are eigenfunctions of the following operators $\left\{\hat{H}, \hat{\Lambda}^2, \hat{T}^2, \hat{K}^2, \hat{T}_3, \hat{T}_{3'}, \hat{K}_3, \hat{K}_{3'}\right\}$, where $\hat{T}_{3'} = \partial/\partial\gamma_T$, $\hat{K}_{3'} = \partial/\partial\gamma_K$, and $\mu,\nu = 0,1,\ldots,7$.

The following equation takes place:

$$\hat{\Lambda}^2\,\Psi^{\mathrm{sph}} = \lambda(\lambda+3)\,\Psi^{\mathrm{sph}}. \tag{19}$$

In the Cartesian coordinates the operator $\hat{\Lambda}^2$ has the form

$$\hat{\Lambda}^2 = -\frac{u^2}{4}\Delta_8 + \frac{1}{4}u_\mu u_\nu \frac{\partial^2}{\partial u_\mu \partial u_\nu} + \frac{7}{4}u_\mu \frac{\partial}{\partial u_\mu}. \tag{20}$$

2. THE 8D CYLINDRICAL BOUND STATES

Let us determine the eight-dimensional cylindrical coordinates as follows:

$$\begin{aligned}
u_0 + iu_1 &= \rho_1 \sin\frac{\beta_T}{2}\,\mathrm{e}^{-i((\alpha_T-\gamma_T)/2)},\\
u_2 + iu_3 &= \rho_1 \cos\frac{\beta_T}{2}\,\mathrm{e}^{i((\alpha_T+\gamma_T)/2)},\\
u_4 + iu_5 &= \rho_2 \sin\frac{\beta_K}{2}\,\mathrm{e}^{i((\alpha_K-\gamma_K)/2)},\\
u_6 + iu_7 &= \rho_2 \cos\frac{\beta_K}{2}\,\mathrm{e}^{-i((\alpha_K+\gamma_K)/2)},
\end{aligned} \tag{21}$$

where $0 \le \rho_1, \rho_2 < \infty$. In these coordinates, the potential, differential elements of the length, volume and Laplace operator have the form

$$V = \frac{\mu_0\omega^2}{2}\left(\rho_1^2 + \rho_2^2\right),$$

$$dl_8^2 = d\rho_1^2 + d\rho_2^2 + \frac{\rho_1^2}{4}dl_T^2 + \frac{\rho_2^2}{4}dl_K^2, \quad dV_8 = \rho_1^3\rho_2^3 d\rho_1 d\rho_2 d\Omega_T d\Omega_K,$$

$$\Delta_8 = \frac{1}{\rho_1^3}\frac{\partial}{\partial\rho_1}\left(\rho_1^3\frac{\partial}{\partial\rho_1}\right) + \frac{1}{\rho_2^3}\frac{\partial}{\partial\rho_2}\left(\rho_2^3\frac{\partial}{\partial\rho_2}\right) - \frac{4}{\rho_1^2}\hat{T}^2 - \frac{4}{\rho_2^2}\hat{K}^2.$$

After the substitution

$$\Psi^{\text{cyl}} = f_1(\rho_1) f_2(\rho_2) D^T_{tt'}(\alpha_T, \beta_T, \gamma_K) D^K_{kk'}(\alpha_K, \beta_K, \gamma_K) \tag{22}$$

the variables in the Schrödinger equation for the eight-dimensional oscillator separated and we arrive at the following system of differential equations

$$\begin{aligned} x_1 \frac{d^2 f_1}{dx_1^2} + 2\frac{df_1}{dx_1} - \left[\frac{T(T+1)}{x_1} + \frac{x_1}{4} - \frac{E_1}{2\hbar\omega}\right] f_1 = 0, \\ x_2 \frac{d^2 f_2}{dx_2^2} + 2\frac{df_2}{dx_2} - \left[\frac{K(K+1)}{x_2} + \frac{x_2}{4} - \frac{E_2}{2\hbar\omega}\right] f_2 = 0, \end{aligned} \tag{23}$$

where $x_i = a^2\rho_i^2$, $a = (\mu_0\omega/\hbar)^{1/2}$, and $E_1 + E_2 = E$. Solutions to Eq. (23) are sought for in the form

$$f_i(x_i) = \mathrm{e}^{-x_i/2}\, x_i^{j_i} W(x_i),$$

where $j_1 = T$, and $j_2 = K$. Then, for $W(x_i)$ we derive an equation for the degenerate hypergeometric function (15) with $\alpha = j_i + 1 - E/2\hbar\omega$ and $\gamma = 2j_i + 2$. Further, introducing cylindrical quantum numbers

$$N_1 = -T - 1 + \frac{E_1}{2\hbar\omega}, \qquad N_2 = -K - 1 + \frac{E_2}{2\hbar\omega}, \tag{24}$$

which are related to the principal quantum number N as follows:

$$N = 2N_1 + 2N_2 + 2T + 2K, \tag{25}$$

normalized by the condition

$$\int |\Psi^{\text{cyl}}|^2\, dV \;=\; 1,$$

the cylindrical basis of the eight-dimensional isotropic oscillator can be written as

$$\begin{aligned} \Psi^{\text{cyl}} = \sqrt{\frac{(2T+1)(2K+1)}{4\pi^4}} \times \\ \times f_{N_1 T}(\rho_1) f_{N_2 K}(\rho_2) D^T_{tt'}(\alpha_T, \beta_T, \gamma_K) D^K_{kk'}(\alpha_K, \beta_K, \gamma_K), \end{aligned} \tag{26}$$

where

$$\begin{aligned} f_{N_i j_i}(\rho_i) = \frac{a^2}{(2j_i+1)!}\sqrt{\frac{2(N_i+2j_i+1)!}{(N_i)!}} \times \\ \times (a\rho_i)^{2j_i}\, \mathrm{e}^{-a^2\rho_i^2/2} F\left(-N_i; 2j_i+2; a^2\rho_i^2\right). \end{aligned} \tag{27}$$

The cylindrical wave functions (26) are the eigenfunctions of both the operators $\left\{\hat{H}, \hat{T}^2, \hat{K}^2, \hat{T}_3, \hat{T}_{3'}, \hat{K}_3, \hat{K}_{3'}\right\}$ and

$$\hat{\mathcal{P}} = \frac{\hbar}{2\mu_0\omega}\left(-\frac{\partial^2}{\partial u_0^2} - \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2} + \frac{\partial^2}{\partial u_5^2} + \frac{\partial^2}{\partial u_6^2} + \frac{\partial^2}{\partial u_7^2}\right) + \\ + \frac{\mu_0\omega}{2\hbar}\left(u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2\right), \quad (28)$$

in this case

$$\hat{\mathcal{P}}\,\Psi^{\mathrm{cyl}} = (2N_1 - 2N_2 + 2T - 2K)\,\Psi^{\mathrm{cyl}}. \quad (29)$$

3. CONNECTION BETWEEN HYPERSPHERICAL AND CYLINDRICAL BASES

At fixed energy values we write down the cylindrical bound states (26) as a coherent quantum mixture of hyperspherical bound states

$$\Psi^{\mathrm{cyl}} = \sum_{\lambda=K+T}^{N/2} W_{NN_1KT}^{\lambda}\Psi^{\mathrm{sph}}. \quad (30)$$

Our goal is the derivation of an explicit form of the coefficients $W_{NN_1KT}^{\lambda}$. First, we should like to note that from the comparison of (10) with (21) we have

$$\rho_1 = u\cos\frac{\theta}{2}, \qquad \rho_2 = u\sin\frac{\theta}{2}. \quad (31)$$

In relation (30), according to (31), we pass from the cylindrical coordinates to hyperspherical ones. Then, substituting $\theta = 0$, taking account of

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

and using the orthogonality condition for radial wave functions in hypermomentum [6]

$$\int\limits_0^\infty R_{N\lambda'}(u)R_{N\lambda}(u)du = \frac{a^2}{2\lambda+3}\delta_{\lambda,\lambda'},$$

we obtain the following integral representation for the coefficients $W_{NN_1KT}^{\lambda}$

$$W_{NN_1KT}^{\lambda} = \frac{\sqrt{(2\lambda+3)(\lambda-K-T)!}}{(2\lambda+3)!(2T+1)!}\,A_{NN_1N_2}^{\lambda KT}\,B_{\lambda KT}^{NN_1}. \quad (32)$$

Here

$$A^{\lambda KT}_{NN_1N_2} = \\ = \left[\frac{(\lambda - K + T + 1)!(N_1 + 2T + 1)!(N_2 + 2K + 1)!\left(\dfrac{N}{2} + \lambda + 3\right)!}{(N_1)!(N_2)!(\lambda + K - T + 1)!(\lambda + K + T + 2)!\left(\dfrac{N}{2} - \lambda\right)} \right]^{1/2}, \quad (33)$$

$$B^{NN_1}_{\lambda KT} = \int_0^\infty x^{\lambda + K + T + 2}\, \mathrm{e}^{-x}\, F\,(-N_1; 2T + 2; x) \times \\ \times F\left(-\frac{N}{2} - \lambda; 2\lambda + 4; x\right) dx, \quad (34)$$

where $x = a^2 u^2$. Further, in (34) writing down the degenerate hypergeometric function $F\,(-N_1; 2T + 2; x)$ as a polynomial, integrating by the formula [7]

$$\int_0^\infty \mathrm{e}^{-\lambda x} x^\nu F(\alpha, \gamma; kx)\, dx = \frac{\Gamma(\nu + 1)}{\lambda^{\nu+1}}\, {}_2F_1\left(\alpha, \nu + 1, \gamma; \frac{k}{\lambda}\right)$$

and taking account of the relation

$${}_2F_1\,(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

we derive

$$B^{NN_1}_{\lambda KT} = \frac{(2\lambda + 3)!(\lambda + K + T + 2)!(N/2 - K - T)!}{(\lambda - K - T)!(N/2 + \lambda + 3)!} \times \\ \times {}_3F_2 \left\{ \begin{matrix} -N_1, -\lambda + K + T, \lambda + K + T + 2 \\ 2T + 2, -N/2 + K + T \end{matrix} \middle| 1 \right\}. \quad (35)$$

Now turning to the integral representation (34) and taking into account (32) and (33), for $W^{\lambda}_{NN_1KT}$ we derive the expression

$$W^{\lambda}_{NN_1KT} = \\ = \left[\frac{(\lambda - K + T + 1)!(\lambda + K + T + 2)!(N_1 + 2T + 1)!(N_2 + 2K + 1)!}{(N_1)!(N_2)!(\lambda - K - T)!(\lambda + K - T + 1)!(N/2 - \lambda)!(N/2 + \lambda + 3)!} \right]^{1/2} \times \\ \times \sqrt{2\lambda + 3}\, \frac{(N/2 - K - T)!}{(2T + 1)!} \times \\ \times {}_3F_2 \left\{ \begin{matrix} -N_1, -\lambda + K + T, \lambda + K + T + 2 \\ 2T + 2, -N/2 + K + T \end{matrix} \middle| 1 \right\}. \quad (36)$$

It is known that the Clebsch–Gordan coefficients can be written as [8]

$$C^{c\gamma}_{a\alpha;b\beta} = \left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2} \times$$

$$\times\, \delta_{\gamma,\alpha+\beta}\frac{(-1)^{a-\alpha}}{\sqrt{(a-\alpha)!}}\frac{(a+b-\gamma)!}{(b-a+\gamma)!}\,{}_3F_2\left\{\begin{matrix} -a+\alpha, c+\gamma+1, -c+\gamma \\ \gamma-a-b, b-a+\gamma+1\end{matrix}\middle| 1\right\}. \qquad (37)$$

Finally, comparing (36) and (37), we arrive at the following representation:

$$W^{\lambda}_{NN_1KT} = (-1)^{N_1} \times$$

$$\times\, C^{\lambda+1,\,K+T+1}_{(N_1+N_2+2K+1)/2,\,(N_2-N_1+2K+1)/2;\,(N_1+N_2+2T+1)/2,\,(N_1-N_2+2T+1)/2}. \qquad (38)$$

The inverse representation has the form

$$\Psi^{\rm sph} = \sum_{N_1=0}^{N/2-K-T} \tilde{W}^{N_1}_{N\lambda KT}\,\Psi^{\rm cyl}. \qquad (39)$$

The expansion coefficients in (39) are given by the expression

$$\tilde{W}^{N_1}_{N\lambda KT} = (-1)^{N_1} \times$$

$$\times C^{\lambda+1,\,K+T+1}_{(N-2T+2K+2)/4,(N-2T+2K+2)/4-N_1;(N+2T-2K+2)/4,N_1+2T-(N+2T-2K-2)/4}. \qquad (40)$$

4. SPHEROIDAL BASIS OF THE 8D OSCILLATOR

Let us determine the eight-dimensional spheroidal coordinates as follows:

$$\begin{aligned}
u_0+iu_1 &= \frac{d}{2}\sqrt{(\xi+1)(1+\eta)}\sin\frac{\beta_T}{2}{\rm e}^{-i(\alpha_T-\gamma_T)/2},\\
u_2+iu_3 &= \frac{d}{2}\sqrt{(\xi+1)(1+\eta)}\cos\frac{\beta_T}{2}{\rm e}^{i(\alpha_T+\gamma_T)/2},\\
u_4+iu_5 &= \frac{d}{2}\sqrt{(\xi-1)(1-\eta)}\sin\frac{\beta_K}{2}{\rm e}^{i(\alpha_K-\gamma_K)/2},\\
u_6+iu_7 &= \frac{d}{2}\sqrt{(\xi-1)(1-\eta)}\cos\frac{\beta_K}{2}{\rm e}^{-i(\alpha_K+\gamma_K)/2},
\end{aligned} \qquad (41)$$

where $\xi \in [1,\infty)$, $\eta \in [-1,1]$, and d is the interfocus distance.

In the spheroidal system of coordinates the oscillator potential has the form

$$V = \frac{\mu_0 d^2 \omega^2}{2}(\xi + \eta).$$

In the coordinates (41), the differential elements of the length, volume and Laplace operator are written in the following form:

$$dl_8^2 = \frac{d^2}{8}(\xi - \eta)\left(\frac{d\xi^2}{\xi^2 - 1} + \frac{d\eta^2}{1 - \eta^2}\right) + \\ + \frac{d^2}{16}(\xi + 1)(1 + \eta) dl_T^2 + \frac{d^2}{16}(\xi - 1)(1 - \eta) dl_K^2,$$

$$dV_8 = \frac{d^8}{512}(\xi - \eta)(\xi^2 - 1)(1 - \eta^2)\, d\xi\, d\eta\, d\Omega_T\, d\Omega_K,$$

$$\Delta_8 = \frac{8}{d^2(\xi - \eta)}\left[\frac{1}{\xi^2 - 1}\frac{\partial}{\partial \xi}\left(\xi^2 - 1\right)^2 \frac{\partial}{\partial \xi} + \frac{1}{1 - \eta^2}\frac{\partial}{\partial \eta}\left(1 - \eta^2\right)^2 \frac{\partial}{\partial \eta}\right] - \\ - \frac{16\hat{T}^2}{d^2(\xi + 1)(1 + \eta)} - \frac{16\hat{K}^2}{d^2(\xi - 1)(1 - \eta)}.$$

After the substitution

$$\Psi^{\text{spheroidal}} = f_1(\xi) f_2(\eta) D^T_{tt'}(\alpha_T, \beta_T, \gamma_K) D^K_{kk'}(\alpha_K, \beta_K, \gamma_K)$$

the variables in the Schrödinger equation separated and we arrive at the following equations:

$$\begin{aligned} &\left[\frac{1}{\xi^2 - 1}\frac{d}{d\xi}\left(\xi^2 - 1\right)^2 \frac{d}{d\xi} + \frac{2T(T+1)}{\xi + 1} - \frac{2K(K+1)}{\xi - 1} + \right. \\ &\qquad \left. + \frac{\mu_0 d^2 E}{4\hbar^2}\xi - \frac{a^4 d^4}{16}(\xi^2 - 1) - X\right] f_1 = 0, \\ &\left[\frac{1}{1 - \eta^2}\frac{d}{d\eta}\left(1 - \eta^2\right)^2 \frac{d}{d\eta} - \frac{2T(T+1)}{1 + \eta} - \frac{2K(K+1)}{1 - \eta} - \right. \\ &\qquad \left. - \frac{\mu_0 d^2 E}{4\hbar^2}\eta - \frac{a^4 d^4}{16}(1 - \eta^2) + X\right] f_2 = 0, \end{aligned} \tag{42}$$

where $X(d)$ is the separation constant in the spheroidal coordinates. Now, excluding energy E from the system of equations (42) we obtain the spheroidal integral of motion

$$\hat{X} = -\frac{1}{\xi-\eta}\left[\frac{\eta}{\xi^2-1}\frac{\partial}{\partial\xi}\left(\xi^2-1\right)^2\frac{\partial}{\partial\xi} + \frac{\xi}{1-\eta^2}\frac{\partial}{\partial\eta}\left(1-\eta^2\right)^2\frac{\partial}{\partial\eta}\right] +$$

$$+\frac{2(\xi+\eta+1)}{(\xi+1)(1+\eta)}\hat{T}^2 - \frac{2(\xi+\eta-1)}{(\xi-1)(1-\eta)}\hat{K}^2 + \frac{a^4d^4}{16}(\xi\eta+1),$$

whose eigenvalues are the spheroidal splitting constant $X(d)$ and the eigenvalues, $\Psi^{\text{spheroidal}}$. Writing down the operator $\hat{X}$ in the Cartesian coordinates we get

$$\hat{X} = \hat{\Lambda}^2 + \frac{a^2d^2}{4}\hat{\mathcal{P}}. \tag{43}$$

Thus, we have

$$\hat{X}\Psi^{\text{spheroidal}} = X_p(d)\Psi^{\text{spheroidal}}, \tag{44}$$

where $0 \le p \le N/2-T-K-1$ numbers the eigenvalues of the operator $\hat{X}$. Now construct the spheroidal basis of the 8D oscillator using the following expansions:

$$\Psi^{\text{spheroidal}} = \sum_{\lambda=T+K}^{N/2} V_{NpKT}^{\lambda}\,\Psi^{\text{sph}}, \tag{45}$$

$$\Psi^{\text{spheroidal}} = \sum_{N_1=0}^{N/2-T-K} U_{NpKT}^{N_1}\,\Psi^{\text{cyl}}. \tag{46}$$

Substituting (45) and (46) into (44), and then using (43) we arrive at the following algebraic equations:

$$\begin{gathered}\frac{4\hbar}{\mu_0\omega d^2}\left[X_p(d)-\lambda(\lambda+3)\right]V_{NpKT}^{\lambda} = \sum_{\lambda'} V_{NpKT}^{\lambda'}\left(\hat{\mathcal{P}}\right)_{\lambda\lambda'}, \\ \left[X_p(d) - \frac{\mu_0\omega d^2}{2\hbar}\left(N_1-N_2+T-K\right)\right]U_{NpKT}^{N_1} = \sum_{N_1{}'} U_{NpKT}^{N_1{}'}\left(\hat{\Lambda}^2\right)_{N_1N_1{}'},\end{gathered} \tag{47}$$

where

$$\left(\hat{\mathcal{P}}\right)_{\lambda\lambda'} = \int \Psi_{\lambda}^{*\text{sph}}\,\hat{\mathcal{P}}\,\Psi_{\lambda'}^{\text{sph}}dV_8, \quad \left(\hat{\Lambda}^2\right)_{N_1N_1{}'} = \int \Psi_{N_1}^{*\text{cyl}}\,\hat{\lambda}^2\Psi_{N_1{}'}^{\text{cyl}}dV_8.$$

Now using expansions (30), (39) and formulae [5]

$$C^{c\gamma}_{a\alpha;b\beta} = -\left[\frac{4c^2(2c+1)(2c-1)}{(c+\gamma)(c-\gamma)(b-a+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^{1/2} \times$$

$$\times\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(b-a+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{4(c-1)^2(2c-3)(2c-1)}\right]^{1/2} \times\right.$$

$$\left.\times C^{c-2,\gamma}_{a\alpha;b\beta} - \frac{(\alpha-\beta)c(c-1) - \gamma a(a+1) + \gamma b(b+1)}{2c(c-1)} C^{c-1,\gamma}_{a\alpha;b\beta}\right\},$$

$$[\,c(c+1) - a(a+1) - b(b+1) - 2\alpha\beta]\, C^{c,\gamma}_{a,\alpha;b,\beta} =$$

$$= \sqrt{(a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1)}\, C^{c,\gamma}_{a,\alpha-1;b,\beta+1} +$$

$$+ \sqrt{(a-\alpha)(a+\alpha+1)(b+\beta)(b-\beta+1)}\, C^{c,\gamma}_{a,\alpha+1;b,\beta-1}$$

and with the orthonormalization conditions [5]

$$\sum_{\alpha+\beta=\gamma} C^{c\gamma}_{a\alpha;b\beta} C^{c'\gamma'}_{a\alpha;b\beta} = \delta_{c'c}\delta_{\gamma'\gamma}, \qquad \sum_{c=|\gamma|}^{a+b} C^{c\gamma}_{a\alpha;b\beta} C^{c\gamma}_{a\alpha';b\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

for the Clebsch–Gordan coefficients of the group $SU(2)$, for the matrix elements $\left(\hat{\mathcal{P}}\right)_{\lambda\lambda'}$ and $\left(\hat{\lambda}^2\right)_{N_1 N_1{}'}$ we get the expressions

$$\begin{aligned} \left(\hat{\mathcal{P}}\right)_{\lambda\lambda'} &= A_{\lambda+1}\delta_{\lambda',\lambda+1} + B_\lambda\delta_{\lambda',\lambda} + A_\lambda\delta_{\lambda',\lambda-1}, \\ \left(\hat{\lambda}^2\right)_{N_1 N_1{}'} &= C_{N_1+1}\delta_{N_1{}',N_1+1} + D_{N_1}\delta_{N_1{}',N_1} + C_{N_1}\delta_{N_1{}',N_1-1}. \end{aligned} \tag{48}$$

Here

$$A_\lambda = -\sqrt{(\lambda-T-K)(\lambda+T+K+2)}\times$$

$$\times\left[\frac{(\lambda+T-K+1)(\lambda-T+K+1)(2N-2\lambda+2)(N+2\lambda+6)}{(\lambda+1)^2(2\lambda+1)(2\lambda+3)}\right]^{1/2},$$

$$B_\lambda = \frac{(N+4)(T-K)(T-K+1)}{(\lambda+1)(\lambda+2)},$$

$$C_{N_1} = -\frac{1}{2}\sqrt{N_1(N_1+2T+1)(N-2N_1-2T+2K+4)(N-2N_1-2T-2K+2)},$$

$$D_{N_1} = N_2(N_1+1)+(N_1+2T+1)(N_2+2K+2)+(T-K)(T-K-1)-2.$$

Substituting expressions (48) into the algebraic equations (47), we derive the three-term recursion relations V^{λ}_{NpKT}, $U^{N_1}_{NpKT}$

$$\begin{gathered} A_{\lambda+1}\,V^{\lambda+1}_{NpKT} + \left\{B_\lambda - \frac{4\hbar}{\mu_0\omega d^2}[X_p(d)-\lambda(\lambda+3)]\right\}\times \\ \times V^{\lambda}_{NpKT} + A_\lambda\,V^{\lambda-1}_{NpKT} = 0, \\ C_{N_1+1}\,U^{N_1+1}_{NpKT} + \left[D_{N_1} - X_p(d) + \frac{\mu_0\omega d^2}{2\hbar}(N_1-N_2+T-K)\right]\times \\ \times U^{N_1}_{NpKT} + C_{N_1}\,U^{N_1-1}_{NpKT} = 0 \end{gathered} \tag{49}$$

for the expansion coefficients V^{λ}_{NpKT} and $U^{N_1}_{NpKT}$. The expansion coefficients V^{λ}_{NpKT} and $U^{N_1}_{NpKT}$ are normalized by the conditions

$$\sum_\lambda \left|V^{\lambda}_{NpKT}\right|^2 = 1, \qquad \sum_{N_1}\left|U^{N_1}_{NpKT}\right|^2 = 1$$

and in the limits $d\to 0$ and $d\to\infty$ turn into

$$\lim_{d\to 0} V^{\lambda}_{NpKT} = \delta_{p\lambda}, \qquad \lim_{d\to\infty} V^{\lambda}_{NpKT} = W^{\lambda}_{NN_1KT},$$
$$\lim_{d\to 0} U^{N_1}_{NpKT} = \tilde{W}^{N_1}_{N\lambda KT}, \quad \lim_{d\to\infty} U^{N_1}_{NpKT} = \delta_{pN_1}.$$

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