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ON SCATTERING OF HYPERGEOMETRIC NATANZON POTENTIALS

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The $so(2,1)$ algebraic treatment for the bound state sector of the hypergeometric Natanzon potentials [1] developed in [2] is extended to include the scattering sector. The formalism introduced in [3] of an asymptotic algebra is used to evaluate the S matrix.

INTRODUCTION

Algebraic techniques have been developed to treat the Schrödinger equation for the bound and scattering sectors, they have been studied many years ago and it is still a living subject. There is a whole class of potentials, the Natanzon potentials [1] for which the complete spectrum has been found in an exact manner by a variety of methods. Among such methods we can mention: a) The SUSYQM approach for shape-invariant potentials [4]; b) The $so(2,1)$ treatment for the confluent Natanzon potentials which has been done some time ago [5]; c) For the Hypergeometric Natanzon Potentials (HNP) there are two algebraic descriptions. One uses an $so(2,2)$ algebra [6], the other one makes use of an $so(2,1)$ algebra [2].

The scattering sector also has been treated in an algebraic way. The Coulomb problem has been treated a long time ago using the $so(3,1)$ algebra to calculate the phase shifts [7], for other approaches of the same problem see [8].

The Pöschl–Teller potential, the reduced case, $V \sim \operatorname{sech}^2(r)$, has been investigated by several authors. In [3], the asymptotic expansion of the $so(2,1)$ algebra was used to obtain the reflection coefficient. This problem was also solved in [9] by means of the Euclidean connection.

The general case for the HNP was solved in [6], after using an $so(2,2)$ algebra together with the Euclidean connection approach.

The technique developed in [9] has been used to analyze the scattering of deformed Coulomb and Pöschl–Teller potentials by means of an $so_q(2,1)$ algebra [11]. More recently, an interesting method has been developed to treat the

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scattering of systems for which the Hamiltonian H is a function of the Casimir operator of an $so(2,1)$ algebra [12].

We first give a brief resume of the algebraic treatment for the bound state sector of the HNP, using an $so(2,1)$ algebra [2]. A specific example, the Rosen–Morse potential, has been treated in order to fix the ideas. Then we analyze the above example for the scattering sector by means of the technique developed in [3]. Finally the general case of the HNP is treated. Therefore, we achieve an algebraic description for the bound and scattering sectors by means of an $so(2,1)$ algebra.

1. BOUND STATE SECTOR, AN EXAMPLE

The hypergeometric Natanzon potentials are given by

$$V = \frac{fz(r)^2 - (h_0 - h_1 + f)z(r) + h_0 + 1}{R(r)} + \\ + \left[a + \frac{a + (c_{1-}c_0)(2z(r) - 1)}{z(r)(z(r) - 1)} - \frac{5}{4}\frac{\Delta}{R(r)}\right]\frac{z(r)^2(1 - z(r))}{R(r)^2}, \quad (1)$$

where the set $\{c_0, c_1, a, h_0, h_1, f\}$ are the Natanzon parameters.

$$\begin{gathered} \tau = c_1 - c_0 - a, \ \Delta = \tau^2 - 4ac_0, \\ R(r) = az(r)^2 + \tau z(r) + c_0, \end{gathered} \quad (2)$$

and the function $z(r)$ satisfies

$$\frac{dz(r)}{dr} = \frac{2z(r)(1 - z(r))}{\sqrt{R(r)}}. \quad (3)$$

The assumptions used for the algrebraization of the HNP, V, by means of an $so(2,1)$ algebra are the following [2]: a) A two-variables (r,ϕ) realization of the algebra; b) The Hamiltonian is related to the Casimir operator of the algebra by

$$(Q - q)\,\Psi(r,\phi) = G(r)(E - H)\Psi(r,\phi), \quad (4)$$

$G(r)$ is a function to be determined by consistency; c) Eigenfunctions of the Casimir operator Q or equivalently, of H, have the form

$$\Psi(r,\phi) = \exp{(im\phi)}\,\Phi(r). \quad (5)$$

The generators are given by

$$J_{\pm} = \exp(\pm i\phi)\left[\pm\frac{\sqrt{z(r)}(z(r)-1)}{z(r)'}\frac{\partial}{\partial r} - \frac{i(1+z(r))}{2\sqrt{z(r)}}\frac{\partial}{\partial\phi} \pm \right.$$
$$\left. \pm\frac{(1-z(r))}{2}\left(\frac{1\mp p}{\sqrt{z(r)}} - \frac{\sqrt{z(r)}z''}{z(r)'^2}\right)\right], \quad J_0 = -i\frac{\partial}{\partial\phi} \qquad (6)$$

with the usual commutation relations: $[J_0, J_{\pm}] = \pm\, J_{\pm}$, $[J_+, J_-] = -2J_0$, where the ladder operators are defined as usual by $J_{\pm} = J_1 \pm iJ_2$. The Casimir operator Q is given by $Q = J_0(J_0+1) - J_-J_+ = J_0(J_0-1) - J_+J_-$, and is found to be

$$Q = (z(r)-1)^2\left[\frac{z(r)}{z(r)'^2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\frac{ip(1+z(r))}{(z(r)-1)\,z(r)}\frac{\partial}{\partial\phi} + \frac{1}{4}\frac{1}{z(r)}\frac{\partial^2}{\partial\phi^2}\right] +$$
$$+ (z(r)-1)^2\left[\frac{1}{2}\frac{z(r)z(r)'''}{z(r)'^3} - \frac{3}{4}\frac{z(r)z(r)''^2}{z(r)'^4} - \frac{1}{4}\frac{p^2-1}{z(r)}\right], \qquad (7)$$

where $z(r)' = (dz(r))/dr$ and p is a function of the Natanzon parameters, independent of $z(r)$. The D^+ representation is used, then the compact operator J_0 is known to have the following eigenvalues

$$m(\nu) = \nu + \frac{1}{2} + \frac{1}{2}\sqrt{4q(\nu)+1}, \; \nu = 0, \dots, \nu_{\max}, \qquad (8)$$

$q(\nu)$ is the eigenvalue of the Casimir operator Q. The energy spectrum is given by

$$2\nu + 1 = \alpha(\nu) - \beta(\nu) - \delta(\nu), \qquad (9)$$

where

$$\begin{aligned} \alpha(\nu) &= \sqrt{-aE+f+1} = p(\nu) + m(\nu), \\ \beta(\nu) &= \sqrt{-c_0E+h_0+1} = p(\nu) - m(\nu), \\ \delta(\nu) &= \sqrt{-c_1E+h_1+1} = \sqrt{4q(\nu)+1}. \end{aligned} \qquad (10)$$

The set $\{p(\nu), m(\nu), q(\nu)\}$ are called the group parameters. It is worthy to note that each solution of the Schrödinger equation belongs to different carrier spaces of the $\{so_{p(\nu)}(2,1)\}$, $\nu = 0, \dots, \nu_{\max}$. The carrier space of the representation given in (6) is

$$\begin{aligned} \Phi(r)_{p(\nu)q(\nu)m(\nu)} &= K\; z^{\beta(\nu)/2}(1-z(r))^{\delta(\nu)/2}R(r)^{1/4} \times \\ &\times\; {}_2F_1(-\nu, \alpha(\nu)-\nu, 1-\beta(\nu), z(r)), \end{aligned} \qquad (11)$$

where K is a normalization constant.

To fix ideas, let us analyze the Rosen–Morse potential defined by

$$V_{\mathrm{RM}} = -2B + 2B \tanh(r) - A(A+1)\,\mathrm{sech}(r)^2, \tag{12}$$

then with the following Natanzon parameters

$$a = 0,\ c_0 = 1,\ c_1 = 1,\ f = 4A(A+1),\ h_0 = -1 - 4B,\ h_1 = -1, \tag{13}$$

together with the function $z(r)$ given by $z(r) = (1 + \tanh(r))/2$, we reproduce (6) after using (1). Notice, if we set $f = 0$ in (13), we obtain the Saxon–Woods potential. The energy spectra are found using (13), (10), and (9), thus one gets

$$E(\nu) = -\frac{(\nu^2 - 2\nu A + A^2 + B)^2}{(\nu - A)^2},\ \nu = 0, \ldots, \nu_{\max},\ \nu_{\max} = \mathrm{intpart}\,(A - \sqrt{-B}), \tag{14}$$

the result for $\nu_{\max}$ is obtained after a careful consideration of ambiguities in the signs occurring in the square roots of equation (10) and using the fact that the energy should increase with ν. Also B is supposed negative in order to support bound states.

The group parameters are obtained from (10) and (13), they are

$$p(\nu) = \frac{2A + 1 + \beta(\nu)}{2},\ m(\nu) = \frac{2A + 1 - \beta(\nu)}{2}, \qquad q(\nu) = -\frac{((\nu - A)^2 + B - A + \nu)}{4(\nu - A)} \tag{15}$$

with

$$\beta(\nu)^2 = -\frac{((\nu - A)^2 + B)^2}{(\nu - A)^2} - 4B.$$

From (6) together with the fact that $z(r) = (1 + \tanh(r))/2$, the generators are then given by

$$J_\pm = \exp(\pm i\phi) \left[\mp \frac{1}{\sqrt{2(1 + \tanh(r))}} \frac{\partial}{\partial r} - \frac{i(3 + \tanh(r)^2)}{2\sqrt{2(1 + \tanh(r))}} \frac{\partial}{\partial \phi} + + \frac{1}{2\sqrt{2(1 + \tanh(r))}} ((p(\nu) \pm 1) \tanh(r) - p(\nu) \pm 1) \right],\quad J_0 = -i\frac{\partial}{\partial \phi}, \tag{16}$$

while the Casimir operator reads as follow

$$Q = \frac{1}{2(1 + \tanh(r))} \left[\frac{\partial^2}{\partial r^2} + \frac{(1 - \tanh(r))^2}{4} \frac{\partial^2}{\partial \phi^2} + + \frac{ip(\tanh(r) - 1)(\tanh(r) + 3)}{2} \frac{\partial}{\partial \phi} - -\frac{1}{4}((\tanh(r)) - 1)^2 (p(\nu)^2 - 1) + 4) \right]. \tag{17}$$

With all this results we have the complete algebraic description for the Rosen–Morse potential. The next step is to study the scattering problem for this example.

2. SCATTERING SECTOR

We have seen in the previous section that the algebra for describing the bound state sector is an $so(2,1)$ one. Following the ideas developed in [3], one can ask for the asymptotic limit, $r \to \infty$, of the bound state algebra given in (16). One can guess that the limiting algebra could be suitable to describe the scattering sector. We are going to see that in fact this is the case. The asymptotic operators are given by

$$J_{\pm}^{\infty} = \exp\left(\pm i\phi\right)\left[-i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\frac{\partial}{\partial r} \pm \frac{1}{2}\right], \quad J_0^{\infty} = -i\frac{\partial}{\partial\phi}, \tag{18}$$

as one can easily verify. The operators given in (19) close in an $so(2,1)$ algebra. Their Casimir operator is

$$Q^{\infty} = \frac{1}{4}\left[\frac{\partial^2}{\partial r^2} - 1\right], \tag{19}$$

we notice that the asymptotic generators obtained are p-independent as one expects. Thus we have the $so(2,1)$ algebra for the bound state sector and also for the asymptotic region.

Let us consider the continuous series of $so(2,1)$ [12]. For this type of representation, the eigenvalues of the Casimir operator, q, are: $q = j(j+1)$ with

$$j = -\frac{1}{2} + i\frac{\lambda}{2},\ \lambda\ \text{real}. \tag{20}$$

The compact generators have eigenvalues given by

$$m = m_0 \pm \sigma,\ \sigma\ \text{integer}. \tag{21}$$

We want to show how the eigenvalues of the Schrödinger equation, E, are related to the eigenvalues of the Casimir operator, q, for the continuum. This can be done by means of the last of the equations given in (10), and we obtain after using (13)

$$E = -4q - 1 = -4j(j+1) - 1 = \lambda^2. \tag{22}$$

The asymptotic states [3, 12] are

$$|j,m\rangle_{\infty} = A_m \exp i(\lambda r + m\phi) + B_m \exp i(-\lambda r + m\phi), \tag{23}$$

where the coefficients or the Jost functions A_m and B_m have to be evaluated. The calculation can be easily carried out if we use the expression for J_{+}^{∞} given in (19) and then act on the asymptotic states (22). The result obtained should be compared with the general expression for the action of generators of an $so(2,1)$ algebra, namely [12]

$$J_{\pm}\left|j,m\right\rangle = \sqrt{(m\mp j)(m\pm j\pm 1)}\left|j,m\pm 1\right\rangle. \tag{24}$$

The following recursion relations are then obtained

$$\begin{aligned} A_{m+1} &= -\frac{1}{2}\frac{(i\lambda - 2m + 1)}{\sqrt{(m-j)(m+j+1)}}A_m, \\ B_{m+1} &= \frac{1}{2}\frac{(i\lambda + 2m + 1)}{\sqrt{(m-j)(m+j+1)}}B_m. \end{aligned} \tag{25}$$

The solution for A_m and B_m are

$$\begin{aligned} A_m &= \frac{\Gamma\Big(-\dfrac{i\lambda}{2} + m + \dfrac{1}{2}\Big)\sqrt{\dfrac{\pi}{\sin\left(\pi(j+1)\right)}}}{\sqrt{\Gamma(m+j+1)\Gamma(m-j)}\,\Gamma\Big(-\dfrac{i\lambda}{2} + \dfrac{1}{2}\Big)}A_0, \\ B_m &= \frac{\Gamma\Big(\dfrac{i\lambda}{2} + m + \dfrac{1}{2}\Big)\sqrt{\dfrac{\pi}{\sin\left(\pi(j+1)\right)}}}{\sqrt{\Gamma(m+j+1)\Gamma(m-j)}\,\Gamma\Big(\dfrac{i\lambda}{2} + \dfrac{1}{2}\Big)}B_0. \end{aligned} \tag{26}$$

Therefore, the reflection coefficient, R_m, defined as $R_m = A_m/B_m$, is then given by

$$R_m = \frac{\Gamma\Big(-\dfrac{i\lambda}{2} + m + \dfrac{1}{2}\Big)\Gamma\Big(\dfrac{i\lambda}{2} + \dfrac{1}{2}\Big)A_0}{\Gamma\Big(\dfrac{i\lambda}{2} + m + \dfrac{1}{2}\Big)\Gamma\Big(-\dfrac{i\lambda}{2} + \dfrac{1}{2}\Big)B_0}. \tag{27}$$

The poles of R_m are located when the following relations hold

$$-\frac{i\lambda}{2} + m + \frac{1}{2} = -\omega, \tag{28}$$

where ω is an integer. Then from (21) and (29) we obtain $j = m + \omega$, and the eigenvalues of the Casimir operator are given by $q = (m+\omega)(m+\omega+1)$. If we use the last equation given in (10) together with (13), we obtain $E = -4q - 1$. Now we use the fact that the group parameter $m = m(\nu) = 1/2(\alpha(\nu) - \beta(\nu))$ as seen in (10). Combining the expression given for E, together with m, we obtain, after using (10) one more time, the following relation

$$E = -(\alpha(\nu) - \beta(\nu) + 2\nu + 1)^2 = -\delta(\nu)^2. \tag{29}$$

This result is consistent with the one given for the general case of the NHP, see (9), up to ambiguities in signs due the occurrence of square roots. In other words, the poles of R_m reproduce the bound state spectra, assuming that A_0 and B_0 are entire functions of λ. This conclude the analysis of the scattering sector for the Rosen–Morse potential.

The next problem is to apply the technique developed for our example to the general case for the NHP. Let us use the Natanzon conditions [1], namely a boundary condition for $z(r)$: $r \to \infty \sim z(r) \to 1$ and demand that V given in (1) should vanish at infinity. After simple calculation one finds the following limit for V

$$V(r \to \infty) = \frac{h_1 + 1}{c_1}, \tag{30}$$

thus $h_1 = -1$ is required. This condition has been applied already in our example, see (13).

A more involved calculation is required for the asymptotic expansion of the generators given in (6), thus one obtains

$$j_{\pm}^{\infty} = \exp\left(\pm i\phi\right)\left[-i\frac{\partial}{\partial\phi} \pm \frac{\sqrt{c_1}}{2}\frac{\partial}{\partial r} \pm \frac{1}{2}\right], \;\; j_0^{\infty} = -i\frac{\partial}{\partial\phi}, \tag{31}$$

those operators clearly close in an $so(2,1)$ algebra, with the following Casimir operator

$$c^{\infty} = \frac{1}{4}\left(c_1\frac{\partial^2}{\partial r^2} - 1\right), \tag{32}$$

as one easily verifies. If we compare, the algebras given in (18) and (31) are the same. In the last one, re scales the variable r. Thus, the results obtained in (27) for R_m are the same.

Let us conclude with a few remarks. First of all, we have achieved an unified treatment for the bound and scattering using an $so(2,1)$ algebra. Second, the approach to the algebraic scattering is simple. Third, the technique of the Euclidean connection for the HNP is a simple application of the work done in [9]. Finally, deformed scattering can be done. Work is in progress using the results given in this paper.

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