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QUANTUM MOTION ALGEBRAS

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The notion of the groups of motions as the groups of elementary changes of quantum states leads to the quantum motion algebras. These algebras can be considered as a good tool for analysis of quantum motion in the many-body problems and also in more fundamental quantum mechanics.

INTRODUCTION

The idea to generate the whole state space of a physical system from a single state, or a family of simple states, was proposed over forty years ago by D. L. Hill, J. A. Wheeler, and J. J. Griffin and named the Generator Coordinate Method (GCM) [1, 2]. It has been very successful in many branches of quantum physics. It is very elegant and powerful method in many-body problems in atomic, molecular and nuclear physics. On the other hand, as is very well known, the symmetries play very important role in both classical and quantum physics. It is also important to notice that in many cases the same group structures are responsible for description of a classical system and its quantum counterpart.

A combination of both ideas, leads to the algebraic generator coordinate method [1–3] useful for description of different kinds of physical problems, e. g., [4–6].

Some preliminary results about the structure of the algebra have been published in [3]. However the physical intuition requires that the group of motion G, responsible for the excitations of the system, should belong to the full algebra of motions. The algebra obtained in [3] does not contain the group G itself because it is a Banach $*$-algebra $L^1(G)$ which cannot contain the unit element [4–7].

The purpose of this introductory paper is to give an outline and extension of the ideas lying behind the notion of the Quantum Motion Algebra (QMA) given in [8].

In addition, we consider a possible quantum origin of the de Broglie relation between the linear momentum and wave vector for relativistic objects to show the other field of applications of the formalism.

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1. GROUP OF MOTION

The classical motion can be described as some changes of states (the generalized positions and the appropriate generalized velocities) along a trajectory in the configuration space. For isolated systems, the evolution operator generates changes from one state (position and velocity) to another one. The family of evolution operators furnish the one parameter group. The parameter is identified with usual time parameter. This idea can be generalized to the notion of the group of motion G as the group responsible for «shifts» of a physical system under consideration from one to another physical state. The classical G motion (the motion generated by the group G) can be related to the appropriate group element $g \in G$. The composition of the group elements $g_1 g_2 \dots g_n$ can be interpreted as a composition of subsequent movements determined by the «shifts» $g_n, g_{n-1}, \dots, g_1$ (just in this order, like the composition of transformations). The inverse element $g^{-1} \in G$ represents an inverse G motion and the neutral element in G can be interpreted as the movement which does not change any physical state of the system.

Though, there are a lot of principal differences between classical and quantum physics, it seems to be possible to extend the idea of the group of motions, defined above, to the quantum world. It seems to be even more appropriate. One can consider the groups which correspond to our geometrical understanding of some motions like space–time translations, space rotations, Lorentzian motions, dilatations and many others. These groups are able to change the states of the physical system (with the extension mentioned in the end of the section) according to our imagination about such kind of motions. In addition, one can consider non-classical types of quantum motions like rotations in the isospin space, the motions generated by different «charges» and other quantities related to symmetries. A good example of the group of quantum motion is also the group of unitary transformations from one to another Slater determinant describing excitations of a set of independent particles. Many well known group theoretical models provide numerous examples of the group of motions understood as the group of transformations among the states of the physical system under consideration.

However, in quantum case the group of motion G is insufficient to describe the full possibilities of quantum motion. In the quantum mechanics, the superposition principle requires to create the whole linear space of allowed states for the system. Because of this we need to consider an algebra of quantum motions containing the group of motion G generating the elementary motions (excitations).

2. ALGEBRA OF QUANTUM MOTIONS

As was mentioned in the previous section, for quantum systems we have no unique path of the motion but the system can choose different paths with some probability amplitudes. In addition, one can observe the linear combinations of the paths. This suggests that the idea of group of motion itself is not sufficient and we should extend the group of motion to the structure which is able to include the superposition principle. The natural extension of the group of motion is to consider the formal sums:

$$\breve{\tau} = \sum_{g\in G} \tau(g)g \tag{1}$$

and the formal integrals [3]

$$\int_G dgu(g)g \tag{2}$$

which are able to describe the interference of elementary motions with amplitudes of motions given either by the discrete function $\tau(g)$ or the function $u(g)$. As can be seen from the definition (1) the group of motion will be included in this structure.

It can be easily shown that instead of rather complicated symbolic integrals (2) it is more convenient to consider the amplitudes of motions $u(g)$ themselves, as in [3].

It means we should consider the algebra of elements of the general form:

$$S = u + \breve{\tau}, \tag{3}$$

where the functions $u \in L^1(G)$ are integrable functions with the norm given by the standard formula [4]:

$$\|u\|_{L^1} = \int_G dg|u(g)| < \infty \tag{4}$$

and the coefficients $\tau(g)$ belong to the space $l^1(G)$ of infinite sequences with the norm [4]:

$$\|\breve{\tau}\|_{l^1} = \sum_{g\in G} |\tau(g)| < \infty. \tag{5}$$

The addition «+» of elements (3) corresponds to quantum interference of motions and it is defined in a natural way. If $S_1 = u_1 + \breve{\tau}_1$ and $S_2 = u_2 + \breve{\tau}_2$, then

$$S_1 + S_2 = u_1 + u_2 + \breve{\tau}, \tag{6}$$

where

$$\breve{\tau} = \breve{\tau}_1 + \breve{\tau}_2 = \sum_{g\in G} (\tau_1(g) + \tau_2(g))g. \tag{7}$$

There exists also the multiplication «∘» of two and more elements (3). One can obtain the explicit form of the multiplication by a composition of two motions (operators) [3]. In principle, it can be viewed as a composition of two «wave packets». This binary operation is associative and bilinear with respect to the addition (interference). Because of this, it is enough to define it between more elementary objects of the algebra than the elements (3).

Namely, for $u, v \in L^1(G)$ and $g, h \in G$ one can write

$$(u \circ v)(g) = \int_G dg' u(g') v({g'}^{-1} g) u \circ g = \Delta_G(g) \mathcal{L}^R_{g^{-1}} ug \circ u = \mathcal{L}^L_{g^{-1}} ug \circ h = gh, \tag{8}$$

where

$$\mathcal{L}^R_{g'} u(g) = u(gg') \mathcal{L}^L_{g'} u(g) = u(g^{-1} g) \tag{9}$$

denotes the right and left shifts on the group manifold and gh the group multiplication, respectively. The function $\Delta_G(g)$ is called the modular function of the group G and it is determined by the right shifts of the left Haar measure [10].

To make unique the definition of the modular function let us denote by dg the left invariant Haar measure for the group G. Then the modular function is defined by the following relation:

$$d(gg') = \Delta_G({g'}^{-1}) dg. \tag{10}$$

Similarly to the first condition in (6) the last one leads also to the convolution operation. Denoting by $\breve{\tau}_k = \sum_{g \in G} \tau_k(g) g$, where $k = 1, 2$, two elements of $l^1(G)$ after using of (8) one obtains

$$\breve{\tau}_1 \circ \breve{\tau}_2 = \sum_{g \in G} \left(\sum_{g' \in G} \tau_1(g') \tau_2({g'}^{-1} g) \right) g. \tag{11}$$

One can show that the set of elements (3) together with both binary operations (6) and (8) (one needs also introduce an obvious operation, multiplication by a number) and the norm

$$\|S\| = \|u\|_{L^1} + \|\breve{\tau}\|_{l^1} \tag{12}$$

furnish the Banach algebra. It consists of two subalgebras $QM'(G)$ which can be identified with the group algebra $L^1(G)$ and the second one $QM''(G)$ which is isomorphic to the group algebra $l^1(G)$. First represents the «continuous sums» of the elementary motions and the second one the discrete series of interfering elementary motions. Formally one can write the $QM(G)$ algebra to be a direct sum:

$$QM(G) = QM'(G) \oplus QM''(G). \tag{13}$$

The subalgebra $QM'(G)$ is two-sided ideal within the algebra $QM(G)$. This is an important property which shows that the composition of «discrete» and «continuous» motions always lead to «continuous» case.

Within the classical mechanics one can easily imagine a notion of inverse motion. This intuition is not obvious for the quantum world, however for elementary motions $g \in G$ the inverse group element g^{-1} represents the inverse quantum motion. In general case one can only think about an analog of inverse motion. More detailed analysis allows one to define an additional operation within the algebra $QM(G)$. In special case of elementary motions it gives an inverse motion, for $QM'(G)$ motions see [9]. This is the unitary involutive operation defined as follows:

$$u^\sharp(g) \equiv \Delta_G(g)u^*(g^{-1}), g^\sharp \equiv g^{-1}, (S^\sharp)^\sharp \equiv S, (\alpha_1 S_1 + \alpha_2 S_2)^\sharp \equiv \alpha_1^* S_1^\sharp + \alpha_2^* S_2^\sharp, \tag{14}$$

where $u \in QM'(G)$, $g \in G$, $S_1, S_2 \in QM(G), \alpha_1$ and α_2 are complex numbers. The involution $\sharp$, in further considerations, plays a role of the operation of Hermitian conjugation in the states space generated by the algebra of motions.

The algebra of quantum motions $QM(G)$ defined above is a combination (nontrivial extension) of well-known, within the theory of locally compact groups representations, the $*$-Banach group algebras $L^1(G)$ and less-known $l^1(G)$ [11]. On the other hand, it can be also found to be isomorphic to a subalgebra of the algebra of measures [9].

The construction described above seems to be the appropriate extension of the notion of the group of motion G to the quantum world.

3. METASTATE AND THE STATES SPACE

In Section 2 we have derived the Banach algebra with involution called the quantum motion algebra, $QM(G)$, which could possibly be able to describe the structure of quantum motion for the system in question. However, the group of motion determines only what kind of motion is under consideration. The question arises what is the physical system which is moving? This can be determined by the metastate known from algebraic approach to quantum mechanics [5, 10].

The metastate is a non-negative, appropriately normalized continuous linear functional on the algebra of motions $QM(G)$. The metastate plays the role of the generator state, known in the generator coordinate method, from which the whole state space is created.

The most general positive and continuous functional can be written in the form ($u \in QM'(G)$ and $\breve{\tau} = \sum_{g\in G} \tau(g)g \in QM''(G)$):

$$\langle \rho; u + \breve{\tau} \rangle = \langle \rho; u \rangle + \langle \rho; \breve{\tau} \rangle, \tag{15}$$

where

$$\langle \rho; u \rangle = \int_{g\in G} dg \langle \rho; g \rangle u(g) \tag{16}$$

and

$$\langle \rho; \check{\tau} \rangle = \sum_{g \in G} \langle \rho; g \rangle \tau(g), \tag{17}$$

where the complex function $\langle \rho; g \rangle$ called the metastate kernel satisfies the following conditions:

(a) $\langle \rho; g \rangle : G \to C$;

(b) $\langle \rho; e \rangle = 1$, where e denotes the unity in G;

(c) $\left\langle \rho; g^{-1} \right\rangle = \langle \rho; g \rangle^{*}$;

(d) for every finite sequences of complex numbers $\alpha_1, \alpha_2, \ldots$, and points on the group manifold $g_1, g_2, \ldots$, the following relation holds

$$\sum_{i,j} \alpha_i^{*} \left\langle \rho; g_i^{-1} g_j \right\rangle \alpha_j \geq 0.$$

In most applications the metastate kernel can be written in quite familiar form

$$\langle \rho; g \rangle = \mathrm{Tr}\,(\rho T(g)), \tag{18}$$

where ρ is a quantum density operator and $T(g)$ denotes a unitary representation of the group G.

Because, it can happen that for a given $z \in QM(G)$

$$\left\langle \rho; z^{\sharp} \star z \right\rangle = 0. \tag{19}$$

To construct the state space one needs to consider the quotient structure

$$QM(G)/\mathcal{R}_{\rho}, \tag{20}$$

where $\mathcal{R}_{\rho}$ consists of elements $z \in QM(G)$ for which $\langle \rho; z^{\sharp} \circ z \rangle = 0$. $\mathcal{R}_{\rho}$ is the left ideal in the algebra $QM(G)$ and one can use the so-called Gelfand–Neumark–Segal construction (GNS) [5]. After standard completion procedure (20) converts into the Hilbert space of states

$$\mathcal{K} \sim QM(G)/\mathcal{R}_{\rho} \tag{21}$$

with the scalar product generated by the metastate:

$$\langle cl_{\mathcal{K}}(S_1) | cl_{\mathcal{K}}(S_2) \rangle \equiv \langle \rho; S_1^{\sharp} \circ S_2 \rangle,$$

where the vector corresponding to $S \in QM(G)$ is denoted $cl_{\mathcal{K}}(S)$ (class of S) or in more traditional form $|S\rangle$.

One needs to notice that in the space of states $\mathcal{K}$ there is the cyclic vector $cl_{\mathcal{K}}(e_G)$

$$cl_{\mathcal{K}}(S) = S\, cl_{\mathcal{K}}(e_G) \tag{22}$$

which allows one to generate all other vectors from the state space $\mathcal{K}$. The element e_G is the unit element in the group of motions G. In other words, it is enough to find the structure of a single state to have immediatelly the structure of all other state vectors. The outline of general construction for the compact groups of motions can be found in [8]. It is much more difficult to find the corresponding construction for noncompact but locally compact groups. This problem requires further extensive investigations.

The construction above is a generalization of the generator coordinate method. Using it one can reproduce all the results of GCM for cases where the generator coordinates can be interpreted as some group parameters. In addition, as generating states one can use here also the density matrices choosing the appropriate form of the metastate kernel, as in (18), what is impossible for traditional GCM. The method used above is also sometimes useful for the construction of irreducible representations of locally compact groups (using instead of $QM(G)$ the corresponding algebra of measures [11]).

4. THE GENERALIZATION OF THE DE BROGLIE RELATION

In the following we consider a nonstandard model using the Poincaré group $G = ISO(1,3)$ as a group of motions. This group is the group of rather complicated global structure and we confine our considerations to the case of spinless, massive particles, i. e., to the scalar representations only.

The elements of the group $G = ISO(1,3)$ can be parametrized by four-translations denoted by the index T, a proper, orthochronous Lorentz transformation which will be denoted by the index L and the usual three-dimensional rotations indicated by R. Using this notation the element of the Poincaré group can be written as:

$$g(x, v, \Omega) = x_T v_L \Omega_R \in ISO(1,3), \tag{23}$$

where x_T is four-translation about the four-vector x; similarly v_L denotes the Lorentz transformation determined by the four-velocity v and finally Ω_R represents the usual rotation about the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

A nearly quite general form of metastate kernel, for scalar representation of massive particle, can be constructed as an average of the standard form of action of the irreducible representations of the Poincaré group:

$$\langle \rho; x_T v_L \Omega_R \rangle = \int_{R^3} \frac{d^3\mathbf{q}}{2q^0} \Psi_0(q)^* \mathrm{e}^{iqx} \Psi_0(\Lambda^{-1}q), \tag{24}$$

where $\Lambda = v_L \Omega_R$, the function $\Psi_0(q)$ we assume to be invariant under the orthogonal group $O(3)$ and $q_0 = \sqrt{\mu^2 + \mathbf{q}^2}$. The Hilbert space of states obtained by GNS procedure with the metastate determined by (24) fulfills the condition:

$$|x_T v_L \Omega'_R\rangle = |x_T v_L \Omega_R\rangle, \tag{25}$$

i. e., the elementary states corresponding to group elements are independent of rotations. Because all other states can be expressed by states (25), we have spinless representation. In this model, the vectors (25) are interpreted as states representing «positions» in the $(10 - 3 = 7)$-dimensional configuration space consisted of points (x, v). More precisely, this configuration space can be constructed in the form of group theoretical orbits of the Poincaré group. To simplify notation, below, instead of v_L the usual symbol for Lorentz transformation Λ is used.

For further purpose we need to define the observable of the wave four-vector. This observable seems to be determined by the spectral family projecting onto representations of the four-dimensional translation subgroup, the Poincaré group:

$$M_k(k) = \left(\frac{1}{2\pi}\right)^4 \int_{R^4} d^4x \, \mathrm{e}^{-ikx} x_T, \tag{26}$$

where $x_T \in T^4 \subset ISO(1,3) \subset QM(G)$. This observable measures the probability of finding in the given state the «space-time periodicity» equal to four-vector k, i. e., the wave properties of the state under consideration. More exactly, the operator (26) is operator-valued distribution and, in principle, we should consider the operators $M_k(\Delta)$ measuring the probability that k belongs to the set $\Delta \subset R^4$. However, we follow the standard procedures in quantum mechanics. Usually, the wave four-vector is identified, by de Broglie relation, with the linear momentum four-vector. The question is if this relation can be derived?

First we write the hermitian operator corresponding to the wave four-vector observable. It can be easily written having in mind the spectral theorem:

$$\hat{k}_\nu = \int_{R^4} d^4k k_\nu M_k(k). \tag{27}$$

Now, let us imagine the particle in the state $|x_T v_L\rangle$. According to the interpretation given above the particle can be viewed as localized in the space-time point x. $\Lambda = v_L$ corresponds to another set of properties which will be clearified in the end of the section.

The probability of finding k in the localized state can be obtained as:

$$\langle x_T \Lambda | M_k(k) | x_T \Lambda \rangle = \left(\frac{1}{2\pi}\right)^4 \int_{R^4} d^4x' \, \mathrm{e}^{-i(\Lambda^{-1}k)x'} \langle \rho; x'_T \rangle . \tag{28}$$

Using the form of the metastate kernel (24) from (28) we get the following formula:

$$\langle x_T \Lambda | M_k(k) | x_T \Lambda \rangle = \delta(\kappa_0 - (\Lambda^{-1}k)_0) \frac{|\Psi_0(\Lambda^{-1}k)|^2}{2\kappa_0}, \tag{29}$$

where $\kappa_0 = \sqrt{\mu^2 + \sum_{l=1}^{3}[(\Lambda^{-1}k)_l]^2}$. The equation (29) allows one to calculate the expectation values of the components of the wave four-vector

$$\langle \hat{k}_\nu \rangle = \int_{R^4} d^4k\, k_\nu \langle x_T \Lambda | M_k(k) | x_T \Lambda \rangle =$$
$$= \int_{R^4} d^4k\, (\Lambda k)_\nu\, \delta\left(\sqrt{\mu^2 + \mathbf{k}^2} - k_0\right) \frac{|\Psi_0(k)|^2}{2k_0}. \quad (30)$$

Because we assumed that the function $\Psi_0(k)$ is invariant under $O(3)$ group we have also that

$$|\Psi_0(k_0, -\mathbf{k})|^2 = |\Psi_0(k_0, +\mathbf{k})|^2 \quad (31)$$

and the integral (30), after the integration over k_0, can be expressed as

$$\langle \hat{k}_\nu \rangle = (\Lambda^{-1})^0{}_\nu \int_{R^3} \frac{d^3\mathbf{k}}{2k_0} \sqrt{\mu^2 + \mathbf{k}^2} |\Psi_0(k_0, \mathbf{k})|^2, \quad (32)$$

where $k_0 = \sqrt{\mu^2 + \mathbf{k}^2}$.

The $\Lambda^0{}_\nu$ coefficients represent the covariant components of the four-velocity components of our particle. This means that the expectation value of the wave four-vector operator $\hat{k}$ for free motion of localized particle is proportional to the velocity four-vector. This is an extension of known de Broglie relation. It is important to notice here that in classical theory the contravariant componets of the wave four-vector are the «physical» ones, i. e., proportional to velocity. Here we are using the covariant componets what leads to the opposite sign of three-velocity. The proportionality coefficient is dependent on structure of the metastate, however, expanding into series the function $\sqrt{\mu^2 + \mathbf{k}^2}$ under the integral (32) we see that

$$\langle \hat{k}_\nu \rangle = (\Lambda^{-1})^0{}_\nu \left\{ \mu + \int_{R^3} \frac{d^3\mathbf{k}}{2\sqrt{\mu^2 + \mathbf{k}^2}} \frac{\mathbf{k}^2}{2\mu} |\Psi_0(k_0, \mathbf{k})|^2 + \ldots \right\}. \quad (33)$$

In the first approximation the integral in (32) is proportional to the invariant mass determined in the metastate kernel. This mass can be also found calculating the expectation values of the square of the wave four-vector operator:

$$\langle \hat{k}^2 \rangle = \langle x_T \Lambda | \hat{k}^2 | x_T \Lambda \rangle = \mu^2. \quad (34)$$

This result allows one to assume that the integral in the formula (32) can be interpreted as the mass of the particle in the localized state $|x_T \Lambda\rangle$.

In this section there has been shown only schematic model of alternative description of space-time relations in relativistic quantum physics. The $QM(G)$ formalism allows one to generate a big variety of models covariant in respect to a given group of motions.

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