ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2002. Т. 33. ВЫП. 7

УДК 539.12.01

AN APPLICATION OF VECTOR COHERENT STATE THEORY TO THE $SO(5)$ PROTON–NEUTRON QUASI-SPIN ALGEBRA

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Vector coherent state theory (VCS), developed for computing Lie group and Lie algebra representations and coupling coefficients, has been used for many groups of interest in actual physical applications. We show that VCS construction of a rotor type can be performed for the $SO(5) \sim Sp(4)$ quasi-spin group where the relevant physical subgroup $SU(2) \times U(1)$ is generated by the isospin operators and the number of particle operators.

Vector coherent state (VCS) theory [1, 2] provides a powerful technique for the evaluation of the matrix representations and the coupling coefficients for Lie algebras of interest in physics. The most commonly used VCS construction involves a boson expansion in terms of a set of Bargmann variables. In a more recent development VCS theory was used for the $SU(3) \supset SO(3)$ algebra chain [3] to generate rotor expansions in terms of standard angular variables. Subsequently, this method was applied to construct irreducible representations (irreps) for the Wigner supermultiplet $SU(4) \supset SU(2) \times SU(2)$ [4] and for $SO(5)$ in an $SO(3)$ basis [5]. The latter case is essential for instance in the classification of multiple quadrupole phonon states in the Bohr model.

The purpose of this paper is to point out that there is another example of a Lie algebra with nuclear structure applications in which a coherent state rotor expansion leads to the explicit expressions for the matrix elements of the generators. This is the algebra of the $SO(5)$ proton–neutron quasi-spin group introduced in connection with the charge-independent pairing problem [6]. The set of the generators of this group consists of the $J = 0$, $T = 1$ pair creation operators $A^\dagger(M_T)$ in the single j shell, the pair annihilation operators $A(M_T)$, the isospin operators $T_\pm, T_0$ and the number of particle operator N. The last four operators span the $U(2) = SU_T(2) \oplus U(1)$ core subalgebra which is of great importance from the physical point of view. It is worth mentioning that the same algebra chain $Sp(4) \sim SO(5) \supset SU(2) \oplus U(1)$ but with different physical meanings of the generators has been recently used in the theory of high T_c superconductivity [7], in the analysis of the spin-mass content of the Bhabha particle [8] and in the discussion of the structure of the most general neutrino mass Hamiltonian [9].

There is another subalgebra which is important for our consideration. It is the direct sum of two commuting $SU(2)$ subalgebras called the neutron and proton quasi-spins and spanned by the sets:

$$I_+ = A^\dagger(1), \qquad J_+ = A^\dagger(-1),$$
$$I_- = A(1), \qquad J_- = A(-1),$$
$$I_0 = \frac{1}{2}(N_0 + T_0), \qquad J_0 = \frac{1}{2}(N_0 - T_0),$$

where $N_0 = (1/2)N - (j + (1/2))$.

The irrep of the $SO(5)$ algebra is conveniently labeled by the highest weight $(\omega_1\omega_2)$, i. e., the eigenvalues of the operators $H_1 \equiv N_0$ and $H_2 \equiv T_0$ when acting on the highest weight state. It can be checked that these labels are related to the seniority v and the reduced isospin t: $\omega_1 = (j + (1/2)) - (v/2)$, $\omega_2 = t$. For brevity, we will denote the irrep labels by (ωt).

The $SO(5) \supset SU_I(2) \oplus SU_J(2)$ subalgebra chain provides four labels I, I_0, J, J_0 to completely specify the basis states within the irrep of the $SO(5)$ algebra. Unfortunately, they are not good quantum numbers. However, when the transformation properties of the basis states under the isospin rotations are used to label the states, one faces the so-called missing label problem because of the occurrence of a nontrivial multiplicity in the generic irrep of $SO(5)$.

We start the construction by taking the set of simple states constructed by adding the proton–proton and neutron–neutron pairs to the state $|vt\rangle$, which by definition does not contain the pairs of coupled nucleons:

$$|p, \alpha\rangle \equiv A^\dagger(-1)^\alpha \, A^\dagger(1)^{p-\alpha} |vt\rangle,$$

where $p = (1/2)(N - v)$ is the number of the nucleon pairs. These states, which we will call intrinsic, have remarkable properties. It is easy to check that the highest weight state is of this form, with $p = 2\omega, \alpha = \omega$. The action of one of six generators $N, T_0, A^\dagger(\pm 1), A(\pm 1)$ on an intrinsic state gives also a state from this set. The result of the action of the operators creating or annihilating neutron–proton pair in the intrinsic state is equal to the result of the action of the isospin laddering operators T_+ or T_- on another intrinsic state with a proper factor:

$$A^\dagger(0)\,|p, \alpha\rangle = \frac{1}{\sqrt{2}(p+1)} T_+ |p+1, \alpha+1\rangle,$$
$$A(0)\,|p, \alpha\rangle = \frac{p-\alpha}{\sqrt{2}} T_- |p-1, \alpha-1\rangle.$$

This property is essential for the construction presented in this contribution. Moreover, every intrinsic state $|p, \alpha\rangle$ is the eigenstate of the operators I^2, I_0, J^2, J_0

and it can be labeled by the following values of the quasi-spin numbers:

$$I = \frac{1}{2}(\omega - t), \qquad J = \frac{1}{2}(\omega + t),$$
$$I_0 = p - \alpha - \frac{1}{2}(\omega - t), \qquad J_0 = \alpha - \frac{1}{2}(\omega + t).$$

It means that the intrinsic states are represented by a corner point on the diagram of admissible I, J values for a given irrep of $SO(5)$.

In general, the intrinsic states do not have a well defined isospin, though it is clearly seen that the third component of the isospin in the state $|p, \alpha\rangle$ has the exact value $K_T = t + p - 2\alpha$ (we assume that the state $|vt\rangle$ has maximal isospin projection t). As has already been shown [10], the complete basis which reduces the $SO(5) \supset SU(2) \oplus U(1)$ subalgebra chain for the generic irrep (ω, t) can be constructed by the Elliott projection technique applied to the states introduced above:

$$|(\omega t)N\alpha TM_T\rangle = \mathbf{P}^T_{M_T K_T} A^\dagger(-1)^\alpha \, A^\dagger(1)^{p-\alpha}|vt\rangle,$$

where α is used as the fourth label to completely label the basis states within the irrep and $\mathbf{P}^T_{M_T K_T}$ denotes the projection operator for the algebra $SU(2)$, which can be taken in the Löwdin–Shapiro form:

$$\mathbf{P}^T_{M_T K_T} = (2T+1)\sqrt{\frac{(T+M_T)!(T+K_T)!}{(T-M_T)!(T-K_T)!}} \times \\ \times \sum_r \frac{(-1)^r}{r!(2T+r+1)!}(T_-)^{r+T-M_T}(T_+)^{r+T-K_T}$$

or in equivalent integral form (the Hill–Wheeler integral):

$$\mathbf{P}^T_{M_T K_T} = \frac{2T+1}{8\pi^2}\int d\Omega \, \mathcal{D}^T_{M_T K_T}(\Omega)\, R(\Omega),$$

where $R(\Omega)$ is the rotation operator with Euler angles Ω in the isospin space and $\mathcal{D}^T_{M_T K_T}(\Omega)$ denotes its matrix element (the Wigner function). Other details concerning the construction, especially the method of choosing the linearly independent elements as the projected states form in fact overcomplete set, can be found in [10]. The general expressions for the overlaps $\langle(\omega t)N\alpha TM_T|(\omega t)N\alpha' TM_T\rangle$ and $\langle(\omega t)II_0JJ_0|(\omega t)N\alpha' TM_T\rangle$ have also been derived.

For the sake of brevity, we will omit henceforth the $SO(5)$ irrep labels (ωt) and instead of $|(1/2)(\omega - t), I_0, (1/2)(\omega + t), J_0\rangle$ for the intrinsic state we will simply write $|I_0 J_0\rangle$. The possibility of projecting the basis means that the rotated states $R(\Omega)|I_0 J_0\rangle$ span the full representation space. According to the general prescription for the construction of the rotor vector coherent state realization

$$|\Psi\rangle \longrightarrow \Psi(\Omega) = \sum_{\rho} |\rho\rangle\langle\rho|R(\Omega)|\Psi\rangle,$$

where ρ stands for the subalgebra irrep labels, a state $|N\alpha T M_T\rangle$ of v unpaired nucleons and $p = (1/2)(N - v)$ nucleon pairs with definite isospin quantum numbers T, M_T is represented by the VCS wave function:

$$\Psi^p_{\alpha T M_T}(\Omega) = \sum_{I_0 J_0} |I_0 J_0\rangle\langle I_0 J_0|R(\Omega)|N\alpha T M_T\rangle,$$

where the summation is restricted to one index by the condition $I_0 + J_0 = p - \omega$. This expression can be transformed to:

$$\sum_{K_T} |I_0 J_0\rangle\langle I_0 J_0|N\alpha T K_T\rangle\langle N\alpha T K_T|R(\Omega)|N\alpha T M_T\rangle,$$

where $I_0 - J_0 = K_T$. The expansion for the VCS function can be finally written as:

$$\Psi^p_{\alpha T M_T}(\Omega) = \sum_{K_T} a^p_{K_T}(\alpha T)\, \mathcal{D}^T_{K_T M_T}(\Omega)\, \xi^p_{K_T},$$

where

$$\begin{aligned} a^p_{K_T}(\alpha T) &\equiv \langle I_0 J_0|N\alpha T K_T\rangle, \\ \mathcal{D}^T_{K_T M_T}(\Omega) &= \langle N\alpha T K_T|R(\Omega)|N\alpha T M_T\rangle, \\ \xi^p_{K_T} &\equiv |I_0 J_0\rangle \end{aligned}$$

with two conditions introduced above: $I_0 + J_0 + \omega = p$, $I_0 - J_0 = K_T$. Thus, we have functions which take vector values in the carrier space for $((1/2)(\omega - t)(1/2)(\omega + t))$ irrep of the subalgebra $SU_I(2) \oplus SU_J(2)$.

The VCS realization $\Gamma(X)$ of any operator X is defined by:

$$[\Gamma(X)\Psi](\Omega) = \sum_{I_0 J_0} |I_0 J_0\rangle\langle I_0 J_0|R(\Omega)|\Psi\rangle,$$

from which follows that the number operator and the isospin operators act in the standard way:

$$\begin{aligned} \Gamma(N)\, \Psi^p_{\alpha T M_T} &= (2p + v)\Psi^p_{\alpha T M_T}, \\ \Gamma(T_0)\, \Psi^p_{\alpha T M_T} &= M_T\, \Psi^p_{\alpha T M_T}, \\ \Gamma(T_\pm)\, \Psi^p_{\alpha T M_T} &= \sqrt{(T \mp M_T)(T \pm M_T + 1)}\, \Psi^p_{\alpha T M_T \pm 1}. \end{aligned}$$

To proceed further in order to obtain the VCS realizations of the generators $A^\dagger(M_T)$ and $A(M_T)$ we note that the triplets

$$\{A^\dagger(1), A^\dagger(0), A^\dagger(-1)\} \text{ and } \{-A(1), A(0), -A(-1)\}$$

transform under isospin rotations as the components of spherical tensors of rank 1:

$$R(\Omega)\,\mathcal{T}^1_\nu\,R(\Omega^{-1}) = \sum_\mu \mathcal{D}^1_{\mu\nu}(\Omega)\,\mathcal{T}^1_\mu .$$

Hence, for any generator $X \equiv \mathcal{T}^1_\nu$:

$$\begin{aligned}
[\Gamma(X)\,\Psi^p_{\alpha T M_T}](\Omega) &= \sum_{K_T} |I_0 J_0\rangle\langle I_0 J_0|R(\Omega)\mathcal{T}^1_\nu R(\Omega^{-1})|N\alpha T K_T\rangle \times \\
&\quad \times \langle N\alpha T K_T|R(\Omega)|N\alpha T M_T\rangle = \\
&= \sum_{\mu, K_T} |I_0 J_0\rangle\langle I_0 J_0|\mathcal{T}^1_\mu|N\alpha T K_T\rangle\,\mathcal{D}^T_{K_T M_T}(\Omega)\,\mathcal{D}^1_{\mu\nu}(\Omega).
\end{aligned}$$

Now it remains to determine the matrix elements $\langle I_0 J_0|\mathcal{T}^1_\mu|N\alpha T K_T\rangle$. Due to the properties of the intrinsic states it is done very easily as $A^\dagger(\pm1)$, $A(\pm1)$ acting to the left produce another intrinsic state, while the generators $A^\dagger(0)$, $A(0)$ have well-defined right action. We get for example:

$$\begin{aligned}
&\langle I_0 J_0|A(1)|N\alpha T K_T\rangle = \\
&\quad = \sqrt{\left(I_0{+}1 - \frac{\omega-t}{2}\right)\left(\frac{\omega+t}{2}+t-I_0-J_0\right)}\langle I_0{+}1, J_0|N\alpha T K_T\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle I_0 J_0|A^\dagger(0)|N\alpha T K_T\rangle = \\
&\quad = -\frac{(I_0-(\omega-t)/2)\sqrt{(T-K_T)(T+K_T)}}{\sqrt{2(J_0-(\omega+t)/2)((\omega-t)/2+\omega-J_0+1)}}\langle I_0, J_0{-}1|N\alpha T, K_T{+}1\rangle.
\end{aligned}$$

Let us recall that the formulas for the overlaps $\langle I_0 J_0|N\alpha T K_T\rangle$ are known. There is a systematic procedure [5] to find the reduced matrix elements. It starts with the expression for $[\Gamma(X)\times\psi^p_T]^{p'}_{T' M'_T}$ and exploits the results presented above but it will not be discussed here.

In conclusion, we have achieved a very simple vector coherent state realization of rotor type for the $SO(5)$ quasi-spin algebra. It is hoped that it will facilitate the evaluation of the coupling coefficients in this case. The complete results of the investigation will be published elsewhere.

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